Using periodics in integer polyhedral problems

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Abstract

Eugene Ehrhart has shown that the number of integer points in a parameterized rational polytope is a particular polynomial whose coefficients are *periodic numbers*. In this research report, we propose a study of these mathematical objects, which depend periodically on parameters, and a generalisation to a class of mathematical objects: the *periodics*. Then, a link between integer points in a polyhedra and the periodics is establshed. Finally, we present a set of polyhedral problems which can be solved using the periodics.

1 Motivation

Integer polyhedral (or linear) problems are numerous in the mathematical literature. In particular, *feasible* solutions in integer linear programming are usually the set of integer points that belong to a polyhedron. In program compilation, a certain kind of frequently used control flow strucures, *loop nests*, allow to repeat the execution of instructions. Each distinct execution, called *iteration*, is indexed by integer variables. These variables are bounded by affine functions and belong to a lattice of integer points. The *polytope model* [1] models these loop nests by the intersection between a polyhedron (defined by the affine bounds) and an integer lattice. This intersection is called a $\mathbb{Z}$-polyhedron. Loop nests can then be optimized and parallelized (e.g. in [8, 6, 9, 5]) by manipulating the corresponding $\mathbb{Z}$-polyhedron.

The work of Eugene Ehrhart [3], Philippe Clauss and Vincent Loechner [2] on counting the integer points in a parameterized rational polytope have brought a particularly interesting structural information. Ehrhart polynomials, which give the number of integer points in a parameterized rational polyhedron, have *periodic numbers* as their coefficients. Periodic numbers are rational numbers depending periodically on parameters. We focus on this periodic aspect, and we want to link it to the object Z-polyhedron.

Frstly, we study the periodic numbers in section 2, as well as a a more general class of mathematical objects, the *periodics*. Then, the link between periodics and integer points in a polyhedron is shown in section 3. The explicit use of this link allows to solve some integer linear problems, which are detailed in section 4. Finally, we present in section 5 an overview of our current and future work.

2 Periodics

In the following, a class of mathematical objects, the *periodics*, which depend periodically on a set of integer numbers, is defined. This class extends the notion of *periodic numbers*, introduced by Ehrhart in the context of counting integer points in a parameterized rational polyhedron. We will then present some specific periodics that will be useful for solving integer linear problems in next sections.

2.1 Definition

Here, we propose two equivalent definitions of the periodics. Each one will be more appropriate for a given use. The following definition presents a periodic as a mapping from integer variables $x \in \mathbb{Z}^n$ to elements of a monoid K. Any distinct reached element of K is periodically reached in function of each variable of x.

Definition 1 *A periodic U of period $S = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \in \mathbb{Z}^n$ over a monoid K is a mapping:*

$$\mathbb{Z}^n \longrightarrow K$$

$$\forall j \in [1, n], x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto u(x) = u(x + t_j),$$

$$\textit{with } t_j = \begin{pmatrix} t_{j1} = 0 \\ \vdots \\ t_{ji-1} = 0 \\ t_{ji} = s_j \\ t_{ji+1} = 0 \\ \vdots \\ t_{jn} = 0 \end{pmatrix}$$

$\square$

A periodic depending on $x \in \mathbb{Z}^n$ will be noted U_x, by convention. n is called the dimension of U. The definition yields the following property:

Property 1 *U_x equals the element u_y of K for all x such that $x = y \bmod S$.*

The elementwise *modulo* operator is used here. We define this operator, which is close to the *division remainder* operator, by:

$$\forall a \in \mathbb{Z}, n, b \in \mathbb{N}, a \textit{ mod } n = b \Leftrightarrow a = k.n + b \textit{ so that } b \geq 0 \textit{ et } k \in \mathbb{Z} \textit{ is maximum}$$

The i^{th} component of $x \textit{ mod } S$ can take s_i distinct values. As the components of S are independent (they are constants), $x \textit{ mod } S$ can take $|S| = s_1 \times s_2 \times \cdots \times s_n$ distinct values. Hence, a periodic can be represented as an array U of dimension n, of size $S = (s_1, s_2, \cdots, s_n)$ as following: $U_x = \mathcal{U}[i_1, i_2, \cdots, i_n]$ if $\forall j \in [1, n]$, $i_j = x_j \textit{ mod } s_j$.

Example 1 A periodic of dimension 1 over $\mathbb{Q}$, of period $S = \begin{pmatrix} 3 \end{pmatrix}$ depending on $P \in \mathbb{Z}$, which equals:

- $\frac{3}{2}$ if P mod $3 = 0$
- 1 if P mod $3 = 1$
- $\frac{7}{8}$ if P mod $3 = 2$

can be represented as: $\begin{bmatrix} \frac{3}{2} & 1 & \frac{7}{8} \end{bmatrix}_P$

A 2-dimensional periodic over affine functions $(i, j, k) \in \mathbb{Z}^n$ dependng periodically on $(i, j) \in \mathbb{Z}^2$, which equals:

- $i + 2j - 3k + 7$ for i mod $3 = 0$, j mod $2 = 0$
- $2i - j + k$ for i mod $3 = 1$, j mod $2 = 0$
- $3j - 5$ for i mod $3 = 2$, j mod $2 = 0$
- $-i - 2j$ +3k -7 for i mod $3 = 0$, j mod $2 = 1$
- $i - 2k + 1$ for i mod $3 = 1$, j mod $2 = 1$
- 0 for i mod $3 = 2$, j mod $2 = 1$

can be represented as:

$$\begin{bmatrix} i + 2j - 3k + 7 & -i - 2j + 3k - 7 \\ 2i - j + k & i - 2k + 1 \\ 3j - 5 & 0 \end{bmatrix}_{i,j}$$

but also as:

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 0 & 0 \end{bmatrix}_{i,j} i + \begin{bmatrix} 2 & -2 \\ -1 & 0 \\ 3 & 0 \end{bmatrix}_{i,j} j + \begin{bmatrix} -3 & 3 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}_{i,j} k + \begin{bmatrix} 7 & -7 \\ 0 & 1 \\ -5 & 0 \end{bmatrix}_{i,j}$$

(this representation is called *factored form*: the generic part of an affine function of (i, j, k) is extracted). □

The property 1 yields a more geometric definition of the periodics. The set of integer points x such that a periodic U_x reaches a given element of K spans a lattice of integer points. Each lattice is then the definition domain of a given element of K reached by U_x.

Definition 2 *A n-dimensional periodic M of period $S = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \in \mathbb{N}^n$ over a monoid K is defined by:*

- *$q = s_1 \times s_2 \times \ldots \times s_n$ elements M_I of K, indexed by:*

$$I = \begin{pmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{pmatrix} \in \mathbb{Z}^n \text{ with } 0 \leq i_k < s_k, k \in [1..n]$$

- *their respective definition domain $\mathcal{D}_I$, which is the integer lattice defined by:*

$$\begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & s_m \end{pmatrix} J + I, J \in \mathbb{Z}^n$$

$\bigcup_I \mathcal{D}_I = \mathbb{Z}^n$: any element $X \in \mathbb{Z}^n$ is associated to a unique M_I by the relation $I = X \bmod S$.

□

2.2 Some interesting periodics

This section enumerates periodics ovec particular monoids. Some of them already exist, the others being useful for solving integer linear problems.

Definition 3 *Periodics over $\mathbb{Q}$ are called* periodic numbers*.*

They have been introduced by Ehrhart [3], where he shows that they are a ring.

Example 2 $\begin{bmatrix} 0 & 2 & 4 \\ 3 & 5 & 1 \end{bmatrix}_{i,j}$ is a periodic number of period $S = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ depending on $\begin{pmatrix} i \\ j \end{pmatrix}$ $\square$

Definition 4 *Periodics over rational affine expressions are called* periodic expressions.

Example 3 $\begin{bmatrix} \frac{1}{3} & 2 \\ -1 & \frac{3}{2} \end{bmatrix}_{N,j} .i + \begin{bmatrix} 0 & 4 & 1 \end{bmatrix}_i .j$ is a periodic expression of period $S = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$ en $\begin{pmatrix} i \\ j \\ N \end{pmatrix}$. $\square$

Definition 5 *Periodics over polyhedra in* $\mathbb{Q}^d, d \in \mathbb{N}$ *are called* periodic polyhedra.

Example 4

$$\begin{cases} \begin{bmatrix} 1 & 2 \\ \frac{-1}{2} & 3 \end{bmatrix}_{N,j} .i + \begin{bmatrix} 0 & 4 & 1 \end{bmatrix}_i .j \geq 0 \\ \begin{bmatrix} 0 & 3 & \frac{2}{5} \\ 1 & \frac{3}{4} & 5 \end{bmatrix}_{i,j} .i + 5 \geq 0 \\ N \geq 0 \end{cases}$$ is a periodic polyhedron. $\square$

According to definition 2, a periodic polyhedron P can be seen as polyhedra P_I with their associated lattice of integer points (their definition domain). Each P_I is then the intersection between a polyhedron and an integer lattice, that is to say a *$\mathbb{Z}$-polyhedron*.

Definition 6 *Introduced by Ehrhart [3] and extended by Clauss and Loechner [2],* Ehrhart polynomial *(or* pseudo-polynomials, arithmetic polynomials *ou* polars*) are periodics over the ring of polynomials of integer variables with rational coefficients.*

Example 5 $\mathcal{E}(i,j,n) = \begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 5 \end{bmatrix}_{i,j} .i^4 j^2 N + \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}_{N,j} .ij^3 + 4N$ is a pseudo-polynomial. $\square$

2.3 Operations on periodics

Notation Periodics over the monoid K will be noted $P[K]$. $\square$

If $\exists k \in [1..n]$ such that $s_k = 1$, we say that the periodic is independent from x_k. Indeed, the element of K reached by U_x is the same for all x_k for given values of $x_1, x_2, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n$.

Any element k of a monoid K is then a periodic U_x over K that equals k for all x. In this section, we give other properties of the periodics, concerning the internal operations. They will be useful to understand the techniques shown hereafter.

The periodics are defined over a monoid K. By definition, a monoid K has an internal operation, that is to say a mapping $K \times K \longrightarrow K$. Let us consider operations on periodics and study them regarding the operations on elements of K.

Theorem 1 *For any operation* C*:*

$$K_A \times K_B \longrightarrow K$$

$$\alpha, \beta \mapsto \mu = C(\alpha, \beta)$$

there exists an operation C':

$$P[K_A] \times P[K_B] \longrightarrow P[K]$$

$$(A_x, B_x) \mapsto U_x = C'(A_x, B_x)$$

for all $(0) \leq x \leq S$*.*

Proof Without loss of generality, assume that the periodics A and B are n-dimensional and of respective periods S^A and S^B. Let $S = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \in \mathbb{Z}^n$ such that $s_i = ppcm(s_i^A, s_i^B)\ \forall i \in [1, n]$. Let $t_i^A \in \mathbb{Z}^n, \forall i \in [1, n]$ of components $t_{i,j}^A = \begin{cases} s_i^A & si\ i = j \\ 0 & sinon \end{cases}$. The vectors t_i^B and t_i are built similarly from the periods S^B and S. Accoring to the definition 2.1 of the periodics, $A_x = A_{(x+t_i^A)}$ and $B_x = B_{(x+t_i^B)}$. As s_i is a multiple of s_i^A and of s_i^B for all $i \in [1, n]$, we also have: $A_x = A_{(x+t_i)}$ and $B_x = B_{(x+t_i)}$. By substitution, the following property is obtained: $C(A_x, B_x) = C(A_{(x+s_i)}, B_{(x+s_i)})$. $C' = C(A_x, B_x)$ is then a periodic of period S. □

According to the theorem 1, C' is a periodic. Moreover, the elements reached by U_x belong to K: U_x is a periodic over K.

Corollary 1 *Periodics* $P[K]$ *over the monoid* K *are a monoid.*

Proof By definition, an internal operation on a monoid K is a mapping $K \times K \longrightarrow K$. According to the theorem 1, there exists for any monoid K with an operation $C : K \times K \longrightarrow K$ a class $P[K]$ with an operation $C' : P[K] \times P[K] \longrightarrow P[K]$. The conservation of the associativity of the operation and of a neutral element is shown easily. □

It yields that periodics over $P[K]$ are periodics over K.

Example 6 Consider the monoid of polyhedra in $\mathbb{Q}^d, d \in \mathbb{N}$ with the operation *intersection* ($\cap$). We have for example:

$$\begin{cases} i + j \geq 0 \\ i \geq -3 \end{cases} \cap \begin{cases} i - j \geq 0 \\ i \leq 10 \end{cases} = \begin{cases} i + j \geq 0 \\ i - j \geq 0 \\ i \leq 10 \end{cases}$$

An operation *intersection* (noted $\cap$) can then be defined from periodic polyhedra to periodic polyhedra. For instance, we have:

$$\left[\begin{cases} i-j\geq 0 \\ i\leq 0 \end{cases} \atop \begin{cases} i+j\geq 0 \\ i\geq -3 \end{cases}\right]_N \cap \left[\begin{cases} i-j\geq 0 \\ i\leq 10 \end{cases} \quad \begin{cases} 2i-j\geq 0 \\ i\geq -3 \end{cases}\right]_M$$

$$= \begin{bmatrix} \begin{cases} i-j\geq 0 \\ i\leq 0 \end{cases} & \begin{cases} -3\leq i\leq 0 \\ 2i-j\geq 0 \end{cases} \\ \begin{cases} i\leq 10 \\ -i\leq j\leq i \end{cases} & \begin{cases} 2i-j\geq 0 \\ i+j\geq 0 \end{cases} \end{bmatrix}_{N,M}$$

□

However, the operation C' defined in the theorem 1 exists, but may not have the same semantic as the operation C. Indeed, according definition 2, the elements of K reached by a periodic $A_x \in P[K]$ are defined on an integer lattice. If the semantic of C depends on the fact that x is on a lattice, then the semantics of C' as defined in theorem 1 is the same as the one of C only if its operands are of period (1).

Example 7 The operation *upper integer part*, $\lceil a \rceil$, is defined for $a \in \mathbb{Q}$ by:

$$\lceil a \rceil = \{n \in \mathbb{Z} \mid n-1 < a \leq n\}, a \in \mathbb{Q}$$

One can deduce, by the theorem 1, an operation with the same definition for $P[\mathbb{Q}]$:

$$upper\_int(U_x) = \{N_x \in P[\mathbb{Z}] \mid N_x - 1 < U_x \leq N_x\}, U_x \in P[\mathbb{Q}]$$

Suppose we want to solve $x = \lceil U_x \rceil = \lceil \begin{bmatrix} \frac{y}{2} & \frac{y}{3} \end{bmatrix}_x \rceil$. One can build the operator $upper\_int(U_x) = \begin{bmatrix} \lceil \frac{y}{2} \rceil & \lceil \frac{y}{3} \rceil \end{bmatrix}_x$. Using the operator *upper_int* to solve the equation would lead to the following result:

$$x = \begin{bmatrix} \{n_0 \mid n_0 - 1 < \frac{y}{2} \leq n_0\} & \{n_1 \mid n_1 - 1 < \frac{y}{3} \leq n_1\} \end{bmatrix}_x$$

For $y = 5$, we would then have $x = \begin{bmatrix} 3 & 2 \end{bmatrix}_x$, that is to say:

$$x = \begin{cases} 3 & for\ x\ even \\ 2 & for\ x\ odd \end{cases}$$

which reduces to $x = \emptyset$. The semantic of the operator *upper_int* is then not to extract the upper integer part of a periodic, as it could be expected. We can see that the semantically corresponding operator would give:

$$\lceil U_x \rceil = \begin{bmatrix} \{2n_0 \mid 2n_0 - 2 < \frac{y}{2} \leq 2n_0\} & \{2n_1 + 1 \mid 2n_1 - 1 < \frac{y}{3} \leq 2n_1 + 1\} \end{bmatrix}_x$$

□

In a more general way, it is always possible to make explicit the value of x in U_x of period p, in order to build an operator that is semantically equivalent for periodics by writing:

$$x = px' + x_0, x_0 = x \ mod \ p$$

In next section, we show a more general interaction between the operators *integer part* and *modulo* and the periodics, in the case of integer linear problems.

3 Integer Parts

We want to model integer linear problems, which can be written as a set of inequalities between (upper or lower) integer parts of rational expressions. In this section, we show the link between this kind of problems and the periodics.

Property 2 *Let $f(I)$ be an integer affine expression of $I = \begin{pmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{pmatrix} \in \mathbb{Z}^n$:*

$$f(I) = c_0 + c_1 i_1 + c_2 i_2 + \cdots + c_n i_n, c_k \in \mathbb{Z}$$

$f(I)$ mod m is a periodic over $\mathbb{Z}$ depending on I of period $B = (b_k)$, where $b_k = \frac{m}{gcd(m,c_k)}, \forall k \in [1, n]$.

Proof Let the vector $I'_k = \begin{pmatrix} i'_1 \\ i'_2 \\ \vdots \\ i'_n \end{pmatrix}$, with $i'_j = \begin{cases} i_j + b_k \ si \ j = k \\ i_j \ else \end{cases}$

$$f(I'_k) = f(I) + c_k b_k$$

But $c_k b_k = \frac{mc_k}{gcd(m,c_k)}$ and $gcd(m, c_k)$ divides c_k. $c_k b_k$ is then an integer multiple of m, so $f(I'_k) \ mod \ m = (f(I) + c_k b_k) \ mod \ m = f(I) \ mod \ m$. Each distinct value of $f(I) \ mod \ m$ is reached periodically depending on i_k, with a period b_k. □

Let a 1-dimensional periodic $A = \begin{bmatrix} a_0 & a_1 & \cdots & a_{m-1} \end{bmatrix}_{f(I)}$ of period m depending periodically on $f(I)$. We have:

$$A = a_j \Longleftrightarrow f(I) \ mod \ m = j$$

with $j \in [0, m-1]$. The values of A are then reached periodically, depending on i_k, with a period b_k.

A can then be represented a periodic of period $S = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$. Note A° this representation. Each value

${a^\circ}_I$ of A° equals the value $a_{f(I)}$ of A. A° is then obtained by scanning each of its entry, which will equal $a_{f(I)}$.

Example 8

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}_{(3i+4j)} = \begin{bmatrix} 0 & 2 & 4 \\ 3 & 5 & 1 \end{bmatrix}_{i,j}$$

□

Property 3 *The period B of the representation A° of A is minimal: there is no periodic depending directly on the indices I with a smaller period (along each component).*

Proof Let q the denominator of $b_k, k \in [1,n]$. By definition, periodics have integer periods: b_k must be integer, and then its denominator q must divide m. Moreover, $c_k b_k$ must be a multiple of m, so q must divide c_k. Hence, the minimal value for b_k is the biggest value of q that divides both m and c_k, that is to say $gcd(m, c_k)$. $b_k = \frac{m}{gcd(m,c_k)}$ is then the smallest possible period of A° along i_k. □

Thus, a 1-dimensional periodic depending on an affine function $f(I)$ from $\mathbb{Z}^n$ to $\mathbb{Z}$ can be represented by a n-dimensional periodic depending directly on I. The inverse operation will be studied later: representing a periodic depending directly on variables by a periodic depending on an affine function of these variables. Generalizing this notion of equivalence to multidimensional periodics depending on affine functions of integer variables could als be useful. However, this operations are not useful for this study, so we will not develop it here.

Now we have defined mathematic objects which will be used for analyzing integer linear problems. A fundamental application is the computation of integer parts of rational affine expressions. It is presented in next section.

3.1 Integer parts of rational expressions

We want to compute the (upper or lower) integer part of an affine function of integer variables I and of parameters N of the form: $f(I,N) = \sum a_k i_k + \sum b_k n_k + c$ avec $a_k, b_k, c \in \mathbb{Q}$. Let q the common denominator of all the coefficients a_k, b_k and c. We can write $f(I,N)$ as following:

$$f(I,N) = \frac{\sum a'_k i_k + \sum b'_k n_k + c'}{q} = \frac{f'(I,N)}{q}$$

with $a'_k, b'_k, c' \in \mathbb{Z}$. Integer lower and upper parts of $f(I,N)$ can be written as:

$$\lfloor f(I,N) \rfloor = \frac{f'(I,N) - (f'(I,N) \ mod \ q)}{q} = f(I,N) - \frac{f'(I,N) \ mod \ q}{q}$$

$$\lceil f(I,N) \rceil = \frac{f'(I,N) + (-f'(I,N) \ mod \ q)}{q} = f(I,N) + \frac{(-f'(I,N)) \ mod \ q}{q}$$

The value of $\lfloor f(I,N) \rfloor$ depends on $f'(I,N)\ mod\ q$, which is periodically depending on $f'(I,N)$. Indeed, $f'(I,N)\ mod\ q$ is the following periodic over $\mathbb{Z}$:

$$f'(I,N)\ mod\ q = \begin{bmatrix} 0 & 1 & \cdots & q-2 & q-1 \end{bmatrix}_{f'(I,N)}$$

We have seen that $f'(I,N)\ mod\ q$ can be written as a periodic over $\mathbb{Z}$:

- of dimension $n = Dim(I) + Dim(N)$,
- depending directly on $x = \begin{pmatrix} N \\ I \end{pmatrix} \in \mathbb{Z}^n$.

It is similar for $\lceil f(I,N) \rceil$.

Example 9

$$\begin{aligned} \lfloor \tfrac{2}{3}i + \tfrac{j}{2} \rfloor & = \lfloor \tfrac{4i+3j}{6} \rfloor \\ & = \tfrac{4i+3j-((4i+3j)\ mod\ 6)}{6} \\ & = \frac{4i+3j-\begin{bmatrix} 0 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}_{i,j}}{6} \\ & = \tfrac{2}{3}i + \tfrac{j}{2} - \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{5}{6} \end{bmatrix}_{i,j} \end{aligned}$$

□

Now the main application for this comes from the fact that a rational polyhedron can be written as a set of equalities and inequalities over rational affine expressions. Many problems reduce to look for an integer point in a rational polyhedron. We present in next section an overview of the problems that are of special interest to us, and that we wish to solve by using periodics.

4 Applications

The application of greatest interest for us is the analysis of loop nest in computer programs. In particular, we focus on loops whose indices are integer and bounded by affine functions of the indices of the enclosing loops and of parameters. This kind of loops constitute a large part of the existing loops. The loop bounds are then integer parts of rational affine expressions.

Example 10 Consider the following loop:

```
for k = (2*j)/3 + i/2 to  7*j + i
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The lower integer bound of k is given by :

$$\begin{array}{rcl} \lceil \frac{2}{3}i + \frac{j}{2} \rceil & = & \lceil \frac{4i+3j}{6} \rceil \\ & = & \frac{4i+3j+((-4i-3j)\ mod\ 6)}{6} \\ & = & \frac{4i+3j+\begin{bmatrix} 0 & 3 \\ 2 & 5 \\ 4 & 1 \end{bmatrix}_{i,j}}{6} \\ & = & \frac{2}{3}i + \frac{i}{2} + \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{5}{6} \\ \frac{2}{3} & \frac{1}{6} \end{bmatrix}_{i,j} \end{array}$$

□

Extracting the integer bounds for a loop nest defines more precisely the values taken by the loop indices. The polytope model models the considered loop nests by rational $\mathbb{Z}$-polyhedra. It is known that the solution of a linear program (when it is unique) is a vertex of the polyhedron P that represents all the possible values for the variables of the problem. Then it is tricky to look for integer solutions inside a polyhedron that has rational vertices. But we can build the polyhedron P' that contains all the integer points of P and whose vertices are integer. This polyhedron, which is the convex hull of the integer points of P, is called *the integer hull* of P. Knowing P' allows to compute the integer lexicographic minimum and maximum of a polyhedron [4], which correspond respectively to the first and last iterations of the loop nest modelized by P. A more general linear solver could also use the integer hull to solve an integer linear programming problem, of the form

$$min\{x \mid f(x, N) \in P\}$$

where x are the variables of the problem, and N are parameters that are in the constant part of the constraints (equalities and inequaities) of P.

Moreover, having the parameterized integer hull of P gives a new method for counting the integer points in P. Indeed, the integer hull of P contains exactly the same number of integer points as P. The number of integer points in P (its Ehrhart polynomial) is then the one of P'.

One can want to know for which value of its parameters a loop nest will be executed or not. This domain is the one for whom there exists an integer point in the associated polyhedron P. We think that this domain of existence of an integer point in P will allow to compute the Ehrhart polynomial of the integer projection D' of a $\mathbb{Z}$-polyhedron D by a projection π. Indeed, it is possible that some integer points of the convex hull of D' are not the image of an integer point in D. The number of integer points in D' is the number of points x of D' whose preimage [1] by π is in D. Taking the coordinates of x as new variables of the problem allow to establish the domain δ for which x exists. The number of integer points in D' is then the Ehrhart polynomial of δ.

[1] The preimage of D' by π is the set of points whose image by π is in D'

5 Conclusion

We have presented the class of periodics, and shown their link to integer linear problems. Using periodics in this context allows an explicit representation of integer parts of rational expressions. This allows to transform rational problems into their integer equivalent. We think that this approach can yield new methods for solving integer linear problems. In particular, we will try to solve the integer lexicographic extrema of a polyhedron, as well as the Ehrhart polynomial of the integer projection of a $\mathbb{Z}$- polyhedron. Their applications are known in program analysis, compilation and optimization (as for example in [8, 6, 9, 5, 10, 7]).

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