Approximations of Polytope Enumerators using Linear Expansions *

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Several scientific problems are represented as sets of linear (or affine) constraints over a set of variables and symbolic constants. When solutions of interest are integers, the number of such integer solutions is generally a meaningful information. Ehrhart polynomials are functions of the symbolic constants that count these solutions. Unfortunately, they have a complex mathematical structure (resembling polynomials, hence the name), making it hard for other tools to manipulate them. Furthermore, their use may imply exponential computational complexity.

This paper presents two contributions towards the useability of Ehrhart polynomials, by showing how to compute the following polynomial functions: an approximation and an upper (and a lower) bound of an Ehrhart polynomial. The computational complexity of this polynomial is less than or equal to that of

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the corresponding Ehrhart polynomial. Their polynomial structure opens the possibility of using them with existing computer algebra systems.

1 Motivation

In this paper, we focus on *rational parametric polyhedra*, defined by a finite set of affine equalities and inequalities with rational coefficients over a set of variables and symbolic constants (that we call *parameters* here). Several research paradigms have shown to rely on counting the number of integer points in such a parametric polyhedron. To our knowledge, so far it has been used in:

- compilation of computer programs, where parts of programs (a class of nested *for / do / while* loops) are represented by polyhedra [2]. Their computational load [27], power consumption, execution time [13, 20, 6, 31, 32], amount of parallelism [32, 10, 5, 23, 16, 15], and memory behavior [30, 9, 18, 22, 14, 17, 1, 7, 36] are then derived from the number of integer points in these polyhedra. These metrics are then used to drive optimizations and parallelization of the compiled program. More examples of Ehrhart polynomials use in the context of embedded systems can be found in [34]. Ehrhart polynomials, initially introduced in the field of program compilation for compilers by Clauss, Loechner and Wilde [8, 11], are now acknowledged as the most relevant way to do it.
- *high-level* embedded systems hardware synthesis [33].
- probability calculations in voting theory [19], where election paradigms as well as cases of interest can be phrased as parametric linear constraints.

Unfortunately, the use of Ehrhart polynomials in production toolchains is hindered by their complex and potentially big-sized data structure, which makes them hard to manipulate. Also, their computational complexity is likely to limit the scalability of these techniques.

In most applications (here, all the above examples except [22, 33]), the precision of Ehrhart polynomials could advantageously be traded for a better ease of manipulation and less computational complexity. Indeed, most of the applications mentionned here would just need an approximate number of integer points in a polyhedron to work. In many cases, the needed result is an upper (or lower) bound on the considered number of points.

In section 3, we propose to approximate Ehrhart polynomials by a rational-valued polynomial. This approximation technique is then extended in section 5 to computing a polynomial which is an upper (or a lower) bound on the Ehrhart polynomial of a polyhedron. Finally, we discuss performance issues and implementation of our algorithms in section 7 and give possible future directions in 8. But first, some prerequisites and notations used across the paper are given in section 2.

2 Prerequisites

This section gives an introduction to Ehrhart polynomials and some notations that we are going to use. Readers who are familiar with Ehrhart polynomials may skip subsection 2.1.

2.1 Ehrhart polynomials and their complexity

A polyhedron is defined by a set of affine constraints (equalities and inequalities) over a set of n variables and *parameters* (symbolic constants). We are interested in the class of problems defined by such constraints, but whose variables take only integer values.

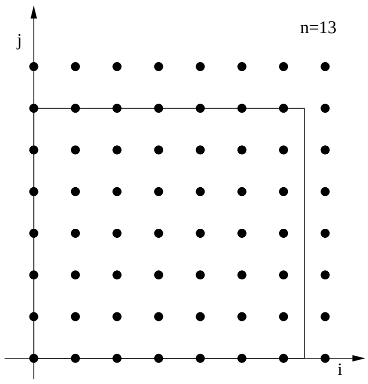

Figure 1: P_1

Example 1. Consider a problem with two variables i, j and one parameter n, with the following set of constraints on (i, j):

$$P_1 = \begin{cases} i \geq 0 \\ j \geq 0 \\ 2i \leq n \\ 2j + 1 \leq n \end{cases}$$

These constraints are represented geometrically, in the 2-dimensional space of the values of (i, j), in figure 1 (here with $n = 13$).

P_1 is empty for any value of $n < 1/2$: its definition domain is $\{n \geq 1/2\}$.

□

The number of distinct integer solutions is often a meaningful peace of information. For instance, in the polyhedral model for nested *for/do* loop nests, this is the number of iterations, that directly relates to the computational load of the loop nest. As the constraints depend on parameters, it is a function of the parameters. Ehrhart showed that this function has a particular analytical form.

Example 2 (1 cont'd)**.** Let us call $\mathcal{E}_{P_1}(n)$ the number of integer values of (i, j) in P_1:

n	1	2	3	4	5	6	7	8	9
$\mathcal{E}_{P_1}(n)$	1	2	4	6	9	12	16	20	25

Ehrhart noticed that the behavior of such a function is not far from that of a polynomial of n. Indeeed, considering odd and even values of n separately, we can define $\mathcal{E}_{P_1}(n)$ using two polynomials:

$$\mathcal{E}_{P_1}(n) = \begin{cases} (\frac{(n+1)}{2})^2 = & \frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{4} \quad ifn\ is\ odd \quad (i.e.,\ n\ mod\ 2 = 1) \\ \frac{n}{2}((\frac{n}{2}) + 1) = & \frac{1}{4}n^2 + \frac{1}{2}n \quad if\ n\ is\ even \quad (i.e.,\ n\ mod\ 2 = 0) \end{cases}$$

As n is periodically odd and even, $\mathcal{E}_{P_1}(n)$ is periodically defined by one of both polynomials: it is a *periodic polynomial*. As only two values of $n\ mod\ 2$ are needed, we can use a more condensed representation for $\mathcal{E}_{P_1}(n)$ using lookup tables for the values of the coefficients:

$$\mathcal{E}_P(n) = \frac{1}{4}n^2 + \frac{1}{2}n + \begin{bmatrix} 0 & \frac{1}{4} \end{bmatrix}_{n\ mod\ 2}$$

The number of dimensions of the lookup table equals the number of parameters of the Ehrhart polynomial.

□

The number of distinct polynomials needed to define an Ehrhart polynomial grows exponentially with the number of bits used to represent the polyhedron. So does then its computational complexity. A more compact way to represent Ehrhart polynomials uses *integer parts* of rational affine functions.[1]

[1]or equivalently, remainders (mod) or fractional parts of affne functions

Example 3. $\mathcal{E}_{P_1}(n)$ can be written:

$$\mathcal{E}_{P_1}(n) = \lceil \frac{n}{2} \rceil \lfloor \frac{n+2}{2} \rfloor,$$

or equivalently:

$$\mathcal{E}_{P_1}(n) = \frac{n + (-n \ mod \ 2)}{2} \frac{n + 2 - ((n+2) \ mod \ 2)}{2} = \frac{n^2 + 2n + (n \ mod \ 2)}{4}$$

□

We will call such a form form the *symbolic form*, by opposition to the form that uses lookup tables, called *explicit form.* Its data structure has only a polynomial complexity in function of the number of bits used to code the problem. Morevover, the algorithm proposed by Barvinok to compute such a form [3], extended and implemented by Verdoolaege et al [34], has also polynomial computational complexity for a fixed number of dimensions. Note that the computational complecity of all the existing algorithms for computing Ehrhart polynomials grows exponentially with the polyhedron's dimension.

Another important applicability issue of Ehrhart polynomials is their use in a toolchain: tools that can manipulate polynomials are available, but none of them can directly deal with periodic polynomials. Very often, the only known way is to use the explicit form[2] and consider each different polynomial in the periodic polynomial separately. This tends to give a de facto exponential complexity to methods that would use Ehrhart polynomials. Moreover, manipulating such structures may turn out to be complex as they involve lookup tables or integer parts functions embedded within a structure encoding multivariate polynomials.

[2]The explicit form can be straightforwardly derived from the symbolic form

2.2 Validity domains

Loechner and Wilde have given an algorithm for partitioning the parameter space of a parametric polyhedron into polyhedral domains within which the polyhedron has a given shape. In each of these domains (called *validity domains*), the number of integer points in the polyhedron is given by a fixed Ehrhart polynomial. The existing algorithms that compute Ehrhart polynomials first compute these validity domains (which are particular cases of *chambers* in polyhedral theory) and provide a different analytical function for each validity domain.

2.3 Lattice notations

Across this paper, some terms related to lattices of integer or rational points will be frequently used. Let X be a rational matrix. $Lat(X)$ denotes the lattice spanned by integer combinations of the column-vectors of X. The *dimension* of a lattice $Lat(X)$ is the dimension of the smallest affine space Y such that $Lat(X) \subset Y$.

2.4 Validity lattices

When some of the constraints of a polyhedron are equalities, it is contained in a linear subspace whose number of spanning vectors is less than the number of variables and parameters, hence it is called *non-full-dimensional*. In this case, the integer points of the polyhedron are included in a lattice, say $Lat(G)$. The variables, $I \in \mathbb{Z}^n$ and the parameters, $N \in \mathbb{Z}^p$, can then be written as:

$$\begin{pmatrix} I \\ N \\ 1 \end{pmatrix} = G. \begin{pmatrix} I' \\ N' \\ 1 \end{pmatrix}, I' \in \mathbb{Z}^n, N' \in \mathbb{Z}^p$$

$Lat(G)$ is called the *validity lattice* [28] of the polyhedron. A upper-triangular G can always be found, and its $(p \times p)$ right bottom sub-matrix of G defines a lattice on the parameters:

$$\begin{pmatrix} N \\ 1 \end{pmatrix} = G'. \begin{pmatrix} N' \\ 1 \end{pmatrix}$$

The polyhedron contains integer points if and only if the parameters belong to $Lat(G')$.

The polyhedron's validity lattice can be used to transform the polyhedron into a new polyhedron whose Ehrhart polynomial equals the Ehrhart polynomial of the original polyhedron[28]. This is done by a compression of a subset of the variables and parameters and a projection of the resulting polyhedron.

Hence we can assume that this pre-processing is done and focus on full-dimensional polyhedra (without equalities) without loss of generality.

3 Using Non-periodicity to approximate Ehrhart polynomials

In this section, we first see in which case a polytope has an Ehrhart polynomial that is just bare, non-periodic, polynomial. We then show that any rational polytope P can be expanded to a polytope P' whose Ehrhart polynomial is not periodic. The used expansion is linear and defines an approximate ratio α between the number of integer points in the expanded polytope and in the original polytope. An approximation of the Ehrhart polynomial of P as a non-periodic polynomial is then straightforwardly given by the Ehrhart polynomial of P' divided by α.

Example 4. Polytope P_1 presented in section 2 can be expanded by

$$\begin{pmatrix} i' \\ j' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix},$$

into polytope

$$P'_1 = \begin{cases} i' \geq 0 \\ j' \geq 0 \\ i' \leq n \\ j' + 1 \leq n \end{cases}$$

The Ehrhart polynomial of P'_1 is

$$\mathcal{E}_{P'_1}(n) = n.(n-1) = n^2 - 2n + 1$$

The approximate ratio between P and P' is 4 (it is the determinant of the expansion matrix), hence:

$$\mathcal{E}_{P'_1}(n)/4 = \frac{n^2 - 2n + 1}{4}$$

is an approximation of $\mathcal{E}_{P_1}$.

□

The expansion from P to P' is derived from a relation, the *non-periodicity condition*, presented in next subsection.

3.1 Non-periodicity Condition

Let P be a rational parametric polytope, i.e. a finite subset of $\mathbb{Q}^n$ defined by a set of m equalities and inequalities with integer coefficients:

$$DI + EN + K \mathrel{\substack{= \\ \geq}} 0, \tag{1}$$

where $I \in \mathbb{Q}^n$ are the variables, $N \in \mathbb{Z}^p$ are the parameters, and D, E, and K are respectively $m \times n$, $m \times p$ and $m \times 1$ matrices with integer coefficients. To understand how to obtain a polyhedron whose Ehrhart polynomial is not periodic, let us review two contributions about mathematical relations between the faces of P and the period of $\mathcal{E}_P(N)$.

The first one is a corollary of Ehrhart's conjecture [12], proved later by Stanley [29] and McMullen [24] and extended by Clauss [8] to the case of several parameters:

Theorem 1. *If all the vertices of P are integer for any integer values of its parameters N, then $\mathcal{E}_P(N)$ is a (non-periodic) polynomial.*

This can be re-formulated by considering that any vertex V of P is defined by turning n of its saturating constraints[3] into equalities $AI + BN + C = 0$. For each vertex, we must then have:

$$\forall N \in \mathbb{Z}^p, AI + BN + C = 0 \Rightarrow I \in Z^n.$$

The second contribution says that the number of integer points in P can be defined by a *linear* recurrence relation on N over a validity domain, which implies that the period of P is independent of the constant part K in (1). This property is also retrieved in [25] (chapter 2): the period of the integer hull of P

[3]a point $I_0(N)$ is said to saturate a constraint $aI + bN + c \mathrel{\substack{= \\ \geq}} 0$ iff $aI_0 + bN + c = 0$

is independent of K, and the Ehrhart polynomial of P equals that of its integer hull.

Looking at the equalities that define a vertex of P, we can see that C is a sub-vector of K. So, as the period of P is independent of K and then from C, we can take $C = 0$, which gives from theorem 1 a less necessary condition for the Ehrhart polynomial of P to be non-periodic:

Theorem 2 (non-periodicity condition)**.** *If, for each vertex of P defined by a set of n equalities $AI + BN + C = 0$, we have:*

$$\forall N \in \mathbb{Z}^p, AI + BN = 0 \Rightarrow I \in Z^n, \tag{2}$$

then its Ehrhart polynomial is a (non-periodic) polynomial.

Linear transformations can be used in different ways to transform a polytope so that it complies to the non-periodicity condition: one can transform the parameters, the variables or both. Here we study a solution that uses a linear transformation of the variables. This choice is discussed later on in section 6.

For each vertex, the solution to (2) is defined by a rational lattice over I:

$$L_v I = I', I' \in \mathbb{Z}^n, \tag{3}$$

where L_v is a square integer matrix. Equation (3) defines an integer linear expansion. Combining the expansions given by all the vertices defines the expansion L by which P will be transformed into a polyhedron P' whose Ehrhart polynomial is non-periodic. Next subsection shows how to compute L_v and subsection 3.3 shows how to combine them to get L.

3.2 Computing the expansion for one vertex

We must find a rational expansion of the variables $I = L_v{}^{-1}I', I' \in \mathbb{Z}^n$ such that $\forall N \in \mathbb{Z}^p, AL_v{}^{-1}I' + BN = 0 \Rightarrow I' \in \mathbb{Z}^n$. Integer values of I' (respectively N) define a regular lattice of points spanned by the column-vectors of matrix $A.L_v{}^{-1}$ (resp.B). For an integer solution I' to exist for each value of N, the point of $Lat(B)$ corresponding to any value of N must be equal to (superimposed with) a point of $Lat(A.L_v{}^{-1})$. In other words, the lattice spanned by the column-vectors of B must be a sub-lattice of the one spanned by those of AL_v^{-1}. We will see in section 4 that the smaller $det(L_v)$, the smaller the approximation error. So we look for the full-dimensional lattice $Lat(AL_v^{-1})$ of greatest determinant of which $Lat(B)$ is a sub-lattice. Let $Lat(E)$ the lattice of biggest determinant of which $Lat(A)$ and $Lat(B)$ are both sub-lattices. $Lat(E)$ is called the *g.c.d.* of A and B. A $n \times n$ integer matrix H that spans such a lattice can be computed by taking the left Hermite normal form of a matrix made by concatenating the column-vectors of A and B:

$$(A\ B) = (H\ 0).U,$$

where U is unimodular. As the equalities defining A and B define a vertex in $\mathbb{Q}^n$, A is square and so is H. The top left $(n \times n)$ sub-matrix U_A of U defines a one-to-one relation between A and H:

$$A = H.U_A$$

As H is the common sub-lattice of B of greatest determinant, U_A is the transformation of smallest determinant that turns $Lat(A)$ into a lattice of which $Lat(B)$ s a sub-lattice.

Hence, U_A is the matrix we are looking for:

$$L_v = U_A.$$

The cost of this expansion is then of a Hermite normal form for each vertex. Next section shows a different method, faster in practice.

3.2.1 A faster expansion

The existing algorithms for computing Ehrhart polynomials have a preliminary step for computing the validity domains computes the parametric coordinates of the vertices. So at this stage, each vertex is defined by the set of n equalities:

$$MI + QN + R = 0,$$

where M is a $n \times n$ diagonal integer matrix. The integer expansion $MI = I'$, satisfies the non-periodicity condition. Its computational cost is low in practice: in the definition of P, the coefficients for variable i_k are just divided by the k^{th} diagonal element of M (it is an *orthogonal* expansion, along the canonical basis vectors). Unfortunately, the determinant of M is not minimal in general. Therefore, the corresponding approximation error is not minimal.

3.3 Non-periodicity for the polyhedron: combining the vertices' expansions

The non-periodicity condition must be satisfied for all the vertices. Note that for any point $I' \in \mathbb{Z}^n$ satisfying the non-periodicity condition, any point I'' resulting from a further integer expansion $I'' = XI'$ also satisfies the non-periodicity condition (as $I' \in \mathbb{Z}^n \Rightarrow I'' \in \mathbb{Z}^n$). Nevertheless, we will see in section 4 that the smaller the expansion, the more accurate the approximation.

Hence, we are looking for an expansion L of minimal determinant that can be written as an expansion of all the vertices' expansions.

Let $I' = L_k I$ be the computed expansion for the k^{th} vertex of P. L must be such that:

$$I = L_k I' = X_k L_k I'' = L I'' \Leftrightarrow L = X_k L_k,$$

where X_k is an integer matrix.

Consider the left Hermite normal form of the matrix made of all the column-vectors of ${L_1}^{-1}, \cdots, {L_q}^{-1}$:[4]

$$\begin{pmatrix} L_1^{-1} & L_2^{-1} & \cdots & L_q^{-1} \end{pmatrix} = \begin{pmatrix} \Lambda & 0 \end{pmatrix} U,$$

where Λ is a full-dimensional $n \times n$ rational matrix and U is an integer unimodular matrix.

We have:

$${L_k}^{-1} = \Lambda U_k \Leftrightarrow \Lambda^{-1} = {U_k}^{-1} L_k, k \in [1..q],$$

where Uk is a $n \times n$ sub-matrix of U. As the U_k's are minimal but integer (as U is unimodular), the expansion matrix L with the smallest determinant resulting from the expansions for the vertices L_q is $L = \Lambda^{-1}$.

Example 5. The following polyhedron:

$$P_3 = \begin{cases} 4i - 6j + 4s - 8t - 1 \geq 0 & (a) \\ -i + 4j + 6t - 3 \geq 0 & (b) \\ -2i + 12s + 8t + 25 \geq 0 & (c) \end{cases},$$

whose variables are i and j and whose parameters are n and m, has three vertices, given by: $v_1 = (a) \cap (b), v_2 = (a) \cap (c), v_3 = (b) \cap (c)$. Each vertex is

[4]Hermite normal forms can be extended to rational matrices by putting them to a common denominator

defined by a system of 2 equalities $AI + BN + C = 0$, so we will note A_1 the matrix of coefficients for the variables defining vertex v_1, and so on.

We have: $L_{v_1}^{-1} = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$, as $A_1 = \begin{pmatrix} 4 & -6 \\ -1 & 4 \end{pmatrix}$ and $B_1 = \begin{pmatrix} 4 & -8 \\ 0 & 6 \end{pmatrix}$. Applying $L_{v_1}^{-1}$ to the variables, one gets an expanded definition of v_1 that respects the non-periodicity condition:

$$\begin{cases} 2i + 4s - 8t - 1 = 0 \\ j + 6t - 3 = 0 \end{cases}$$

Similarly, we have:

$$A_2 = \begin{pmatrix} 4 & -6 \\ -2 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 4 & -8 \\ 12 & 8 \end{pmatrix} \Rightarrow L_{v_2}^{-1} = \begin{pmatrix} 0 & -1 \\ -\frac{1}{3} & -\frac{2}{3} \end{pmatrix},$$

giving:

$$\begin{cases} 2i + 4s - 8t - 1 = 0 \\ 2j + 12s + 8t + 25 = 0 \end{cases}.$$

Finally,

$$A_3 = \begin{pmatrix} -1 & 4 \\ -2 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 6 \\ 12 & 8 \end{pmatrix} \Rightarrow L_{v_3}^{-1} = \begin{pmatrix} -1 & -2 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

giving:

$$\begin{cases} i + 6t - 3 = 0 \\ 2i + 4j + 12s + 8t + 25 = 0 \end{cases}.$$

The g.c.d. of the lattices $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ spanned respectively by the column-vectors of L_1^{-1}, L_2^{-1} and L_3^{-1} is spanned by the column-vectors of L^{-1}:

$$L^{-1} = \begin{pmatrix} \frac{1}{5} & 0 \\ \frac{2}{15} & \frac{1}{6} \end{pmatrix}$$

The expansion for which the non-periodicity condition is satisfied for all the vertices of P_3 is then given by

$$\begin{pmatrix} i \\ j \end{pmatrix} = L^{-1} \begin{pmatrix} i' \\ j' \end{pmatrix},$$

and the resulting polyhedron P_3' with an expanded variable space is:

$$P_3' = \begin{cases} -j' + 4s - 8t - 1 \geq 0 \\ i' + 2j' + 18t - 9 \geq 0 \\ -2i' + 60s + 40t + 125 \geq 0 \end{cases}.$$

The number of integer points in P_3, given by its Ehrhart polynomial, is:

$$\mathcal{E}_{P_3}(s,t) = \frac{361}{30}s^2 + \frac{209}{15}st + \frac{121}{30}t^2 + \frac{1007}{30}s + \frac{583}{30}t +$$

$$
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\end{bmatrix}_{s \ mod \ 15, t \ mod \ 15}
$$

Its approximation, given by the Ehrhart polynomial of the expanded polyhedron P'_3 divided by $det(L) = 30$ is:

$$\mathcal{E}_{P_3}(s,t) \simeq \frac{361}{30}s^2 + \frac{209}{15}st + \frac{121}{30}t^2 + \frac{1007}{30}s + \frac{583}{30}t + \frac{117}{5}.$$

The absolute error δ with the original Ehrhart polynomial lies within a constant interval: $0 \leq \delta \leq \frac{4}{10}$. In comparison, using the approach of section 3.2.1 we get:

$$L_{v_1}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}, L_{v_2}^{-1} = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}, L_{v_3}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

and expansion matrix $L^{-1} = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{30} \end{pmatrix}$ with $det(L) = 150$, which gives a less precise approximation of P_3:

$$\mathcal{E}_{P_3}(s,t) \simeq \frac{361}{30}s^2 + \frac{209}{15}st + \frac{121}{30}t^2 + \frac{4921}{150}s + \frac{2849}{150}t + \frac{559}{25}.$$

whose absolute error δ is characterized by: $\frac{19}{25}s + \frac{11}{25}t + \frac{16}{25} \leq \delta \leq \frac{19}{25} + \frac{11}{25}t\delta + \frac{103}{75}$. As its non-constant part is linear, we can obtain its extremal values by evaluating it at P's extremal values for (s,t) and see that the absolute error is unbounded over P's existence domain.

□

3.3.1 Another tradeoff: expanding per validity domain

Existing algorithms for computing Ehrhart polynomials first compute all the vertices, deriving validity domains in which a fixed subset of the vertices is not redundant in P. One can compute an expansion per validity domain and work only on non-redundant vertices, or compute a global expansion that takes into account all the vertices that are *eventually* non-redundant. As the polyhedra share vertices across different validity domains, the former method may imply redundant computations of vertex expansions. However, each individual expansion would concern fewer vertices. As a consequence, each expansion matrix (L) would have a smaller determinant within each validity domain, so the approximation would be more accurate. However, as there can easily be many validity domains, it is likely that the latter method gives better performance if the redundant computation of vertices is not avoided. Indeed these options already appear when computing Ehrhart polynomials, so we may consider it as independent of the approximation problem itself.

4 Approximation error

In example 5, notice that all the terms of degree 2 (i.e., the terms for which the sum of the degrees along each variables equals 2) of the approximation $\mathcal{E}_{P_3'}(N)$ equal the terms of degree 2 of the original Ehrhart polynomial. In this section, we show that it is always the case for a full-dimensional polyhedron.

Theorem 3. *Let $\mathcal{E}(N)$ be the Ehrhart polynomial of a n-dimensional polyhedron P. The terms of degree n of the approximation $\mathcal{E}'(N)$ by variable expansion equal the terms of degree n of $\mathcal{E}(N)$.*

A proof of this theorem is given in [25]. We just give a sketch of this proof here, which will also be useful for understanding how to compute a polynomial upper/lower bound of an Ehrhart polynomial.

The variable space $\mathcal{S}$ can be paved into *unit cells* defined by

$$\mathcal{C}(I_0) = \{I \mid I_0 \leq I < I_0 + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\}, I_0 \in \mathbb{Z}^n$$

where the $\leq$ and $<$ operators are element-wise. Each unit cell contains exactly one integer point. When $\mathcal{S}$ is expanded by the transformation $I = L^{-1}I'$, giving the expanded space $\mathcal{S}'$, each unit cell is transformed into an *expanded cell* given by:

$$\mathcal{C}'(I_0) = \{I' \mid I_0 \leq L^{-1}I' < I_0 + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\},$$

which contains exactly $g = det(L)$ integer points.

If the polyhedron P could be partitioned exactly into whole unit cells (i.e., if it is a hyper-rectangle with integer vertices), then its image by expansion P'

would be the union of the image of the unit cells of P.

Then, its number of integer points would be exactly g times the number of integer points in P. But in the general case, some unit cells are partly inside P and partly outside. Two cases can be distinguished:

- a unit cell with $I_0 \notin P$ but having a non-empty intersection with P: the corresponding *expanded cell* may contain integer points that belong to P. As these integer points do not correspond to an integer point of P, they are *extra points* which tend to lead to an over-approximation of $\mathcal{E}(N)$.

- conversely, a unit cell with $I_0 \in P$ but not being completely included in P: the corresponding *expanded cell* may contain integer points that do not belong to P. As these integer points do correspond to an integer point of P, they are *missing points* which tend to lead to an under-approximation of $\mathcal{E}(N)$.

Example 6. The unit cell $\mathcal{C}(0,0) = \{0 \leq i < 1; 0 \leq j < 1\}$ and the polyhedron $\{3i + 4j - 2 \geq 0\}$ in $\mathbb{Z}^2$ are represented in figure 2. They have a non-empty intersection. Expanding the variable space by $\begin{pmatrix} i \\ j \end{pmatrix} = L^{-1} \begin{pmatrix} i' \\ j' \end{pmatrix}$ with $L = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ gives the expanded unit cell $\mathcal{C}'(0,0) = \{0 \leq i'+j' < 2; 0 \leq i'-j' < 2\}$ and the expanded polyhedron $P' = \{7i' - j' - 4 \geq 0\}$ represented in figure 3.

□

We then have:

$$\mathcal{E}(N) = \mathcal{E}'(N) + (nmp(N) - nep(N))/g,$$

where *nmp* and *nep* stand respectively for *number of missing points* and *number*

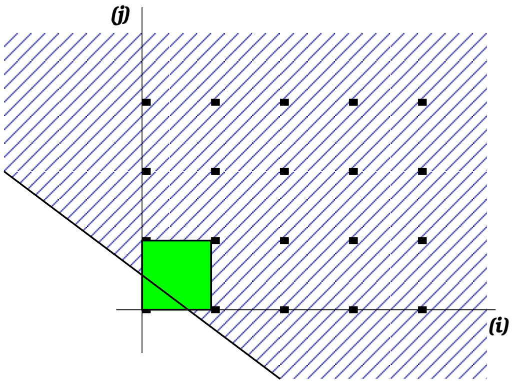

Figure 2: A unit cell having a non-empty intersection with a polyhedron

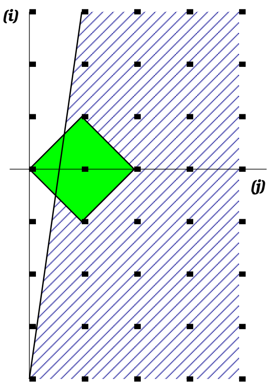

Figure 3: The expanded unit cell and polyhedron, with one extra integer point

of extra points. The approximation error ξ is then given by:

$$\xi = |(nmp(N) - nep(N))/g| \qquad (4)$$

The proof in [25] shows that this difference is given by the number of integer points contained in a finite number of $n-1$-dimensional polyhedra, which is an Ehrhart polynomial of degree $n-1$. This number of $n-1$-dimensional polyhedra increases with the determinant of the expansion, along with their number of integer points, and then the approximation error. As the difference between $\mathcal{E}(N)$ and $\mathcal{E}'(N)$ is of degree $n-1$, the terms of $\mathcal{E}'(N)$ of degree n are the same as those of $\mathcal{E}(N)$.

The consequence is that the relative approximation error $\mathcal{E}(N)/\mathcal{E}'(N)$ asymptotically tends towards zero.

5 Upper/lower bounds as a polynomial

We have seen that expanding the variable space of P introduces extra points, which are responsible for over-approximating $\mathcal{E}_P(N)$, and missing points, which induce an under-estimation of $\mathcal{E}_P(N)$.

Here, we look for a polynomial that gives an upper bound on $\mathcal{E}_P(N)$. Let P' be the polyhedron whose Ehrhart polynomial is an approximation of $\mathcal{E}_P(N)$ by variable expansion. The idea is to compute a polyhedron P_{upper} whose expansion P'_{upper} will contain all the missing points of P' (in addition to the extra points): its approximate Ehrhart polynomial will then be greater than $\mathcal{E}_P(N)$.

5.1 Tight inflation

Basically, any rational point $I_0 \in P$ must have its whole unit cell in P_{upper}:

$$\forall q \in [0,1)^n, I_0 \in P \Rightarrow I_0 + q \in P_{upper} \tag{5}$$

There is a simple way to compute P_{upper} from P: we can get P_{upper} from P by *inflating* it, i.e., translating its borders outwards by a constant so that it satisfies (5). We can take each inequality $f : aI + bN + c \geq 0$ of P, and *inflate* it into an inequality for f_{upper}: $aI + bN + c' \geq 0$, with $c' \geq c$. Let us define $c' = c + c_a, c_a \geq 0$. As we want the over-approximation to be as tight as possible, we are looking for the minimal value of c_a so that if an integer point I_0 satisfies f, any missing point in I_0's unit cell satisfies f_{upper}:

$$\forall q \in [0,1)^n, aI_0 + bN + c \geq 0 \Rightarrow a(I_0 + q) + bN + c' \geq 0, \tag{6}$$

where q takes values corresponding to rational points that will be expanded into integer points. A sufficient condition for $a(I_0 + q) + bN + c' \geq 0$, i.e., $aI_0 + bN + c + aq + c_a \geq 0$ to be implied by $aI_0 + bN + c \geq 0$ is that $aq + c_a \geq 0$, i.e.,

$$c_a \geq -aq \tag{7}$$

The maximum value for $-aq$ is obtained when the coordinates of q corresponding to negative elements of a are set to the greatest rational value that is less than one and the others coordinates set to zero. Hence, the minimal c_a necessary for (6) to be true is given by minus the sum of the negative elements of a:

$$c_a = -\sum_{\{k | a_k < 0\}} a_k - \nu,$$

and $\nu > 0$ is a rational that has to be small enough for f_{upper} to include all the rational points that will be expanded into integer missing points.

Hence, f_{upper} is defined as

$$aI + bN + c - \sum_{\{k|a_k<0\}} a_k - \nu \geq 0. \tag{8}$$

We can distinguish two cases, depending on our knowledge of the expansion to be applied at the time we inflate P.

5.2 When the expansion is not know yet

In this case, there is no way to know which rational points of P will become missing points once expanded. Hence, we have to conservatively assume that the determinant of the expansion matrix is infinite, which corresponds to $\nu \to 0^+$:

$$f_{upper} : aI + bN + c - \sum_{\{k|a_k<0\}} a_k > 0$$

When the underlying computing system assumes that the variables are integer, it may turns strict inequalities into non-strict ones by using the rule

$$x \in \mathbb{Z} \Rightarrow (x > 0 \Leftrightarrow x - 1 \geq 0). \tag{9}$$

Doing so would yield a wrong result, as we are implicitly considering rational points that will become integer after expansion. In practice we just take $\nu = 0$ and we have:

$$f_{upper} : aI + bN + c - c_a \geq 0$$

5.3 When the expansion is known

When the expansion to be applied is known at the time we want to inflate the polyhedron ($I = L^{-1}I'$), we can compute the maximum value for $-aq$ in (7) with $q \in [0,1)^n$ and $q = L^{-1}q', q' \in \mathbb{Z}^n$, which defines an integer linear programming problem. As a simpler heuristic, we may choose the smallest ν so that P_{upper} contains a potential missing point. The heuristic is that the points on $Lat(L)$ that are closest to f_{upper} but not on it has good chances of being a missing point.

f_{upper} is expanded into f'_{upper} :

$$aL^{-1}I' + bN + c - \sum_{\{k|a_k<0\}} a_k - \nu \geq 0.$$

Multiplied by the denominator d of its coefficients except ν, it gives the following inequality with integer coefficients:

$$da'I' + dbN + dc - \sum_{\{k|a_k<0\}} da_k - d\nu \geq 0$$

It is well known that for $da.L^{-1}I' + dbN + dc - \sum_{\{k|a_k<0\}} da_k - d\nu = 0$ to have an integer solution, $dc - \sum_{\{k|a_k<0\}} da_k - d\nu$ must be a multiple of the g.c.d. g of the coefficients of $da.L^{-1}$ and db. This gives:

$$(dc - \sum_{\{k|a_k<0\}} da_k - d\nu) \ mod \ g = 0.$$

As, by definition, d divides g, it gives:

$$(c - \sum_{\{k|a_k<0\}} a_k - \nu) \ mod \ \frac{g}{d} = 0$$

Also, we do not want to make f_upper tighter than f. As we want the smallest

ν, we take $0 < d\nu \leq g$ and get:

$$\nu = \begin{cases} 0 & if \ \sum_{\{k|a_k<0\}} a_k = 0 \\ \frac{g}{d} & if \ (c - \sum_{\{k|a_k<0\}} a_k) \ mod \ \frac{g}{d} = 0 \\ (c - \sum_{\{k|a_k<0\}} a_k) \ mod \ \frac{g}{d} & else \end{cases}$$

P_{upper} is constructed by inflating all its inequalities this way. The polynomial upper bound of $\mathcal{E}_P(N)$ is then given by the approximation of the Ehrhart polynomial of P_{upper}, $\mathcal{E}_{P'_{upper}}(N)$.

Similarly, a polynomial lower bound can be computed by taking the approximation of the Ehrhart polynomial of a *deflated* polyhedron P_{lower}, which does not contain any extra points. This is derived by transforming the inequalities of P: $aI + bN - c \geq 0$ into $aI + bN + c - \sum_{\{k|a_k>0\}} a_k + \nu \geq 0$.

5.4 Lower bound: degenerate cases

Deflating a polyhedron may result in an empty polyhedron. This can be a problem in some degenerate cases, in which the deflated polyhedron is *always* empty even though the original polyhedron has a variable and possibly big number of points.

In this section, we try to characterize these cases, leaving the question of how to deal with them for future work. The lower bound obtained in this case is zero, which is still less than the actual Ehrhart polynomial of the polyhedron. However, it is quite unsatisfying and one may wish to obtain a more accurate approximation.

The class of pathological cases that we identified is made of prisms that are *too tight* for any value of the parameters, intersected with other constraints. Let us try to characterize the constraints that make such a prism. For simplicity, let us consider the full-dimensional polyhedron P' made of m inequalities of P

involving its n variables I, and involving or not its p parameters N:

$$P : AI + BN + C \geq 0, A \in \mathbb{Z}^{m \times n}, B \in \mathbb{Z}^{m \times p}, C \in \mathbb{Z}^m$$

P' is *too tight* if it is characterized by two properties:

a deflating P''s constraints gives an empty polyhedron for some value of the parameters;

b modifiying the parameter's values only results in a translation of P'.

c P' has q *lines* i.e., it can be written

$$P' = P'' + \sum_{k=1}^{q} \alpha_k l_k, l_k \in \mathbb{Z}^{n+p}, \forall \alpha_k \in \mathbb{Q}, \tag{10}$$

Property (a) can be tested directly by deflating P'. Property (b) boils down to

$$r(\left(A \mid B \right)) = r(A), \tag{11}$$

where $r(A)$ denotes the row-rank of matrix A. (c) can be written:

$$r(A) < n. \tag{12}$$

Example 7. In polytope $P2(s, t)$:

$$\begin{cases} i - 2j + s + 1 \geq 0 \\ -i + 2j - s \geq 0 \\ i - j \geq 0 \\ -i + t \geq 0 \end{cases}$$

the two first inequalities have arow-rank of 1, as well as their restriction to the variables. As it is a 2-dimensional polyhedron, it is *degenerate*. Applying tdeflation ot this polyhedron gives a lower bound of zero for $\mathcal{E}_{P_2}(s,t)$.

□

6 Other approximation methods

6.1 Rounding bounds of nested intervals

Tawbi [31] expresses the number of integer points in a polyhedron as nested sums which are then decomposed using Bernoulli formulae. The main drawback of this method is that it has to split the polyhedron so that it can be expressed as the union of polyhedra that can be written as nested intervals. This also implies splitting the validity domains further on. Therefore, it is more complex than the existing Ehrhart polynomials algorithms. When the vertices of the obtained polyhedra are not integer, an approximation (a mean value, seemingly) of the difference between the rational bounds and their corresponding integer bound is added to (or substracted from) the nested intervals.

This leads to an approximation of the number of integer points in each of the split domains. This method is likely to bring an approximation that evolves similarly to the Ehrhart polynomial.

6.2 Interpolation by a polynomial

In the context of computing the worst case execution time of programs, Van Engelen, Gallivan and Walsh [13] compute a polynomial for approximating the number of integer points in a parametric polyhedron by interpolation (over values of the parameters N).

This method might also handle polynomial loop bounds, in which case it generally leads to an approximation of the number of iterations. Unfortunately, it does so dimension by dimension and hence has to split the polyhedron into nested intervals, as in [31]. As each step works on a one-dimensional parametric interval, the upper bound is computed by adding *one* to each interval. This is because the number of integer points in an interval is greater than the rational lenght of the interval minus one. A lower bound could be obtained similarly.

6.3 Another interpolation by a polynomial

It has been shown [25] that the Ehrhart polynomial of P is a polynomial if the values of the parameters N are restricted to a certain lattice (in which the non-periodicity condition is fulfilled for P). Such a polynomial gives the exact value of the Ehrhart polynomial of P when N is on the given lattice. This guarantees that the obtained polynomial evolves like the Ehrhart polynomial of P. The space of parameters can be compressed to restrict the integer values of the new parameters correspond to values of the original parameters that respect the non-periodicity condition. The Ehrhart polynomial of this compressed polyhedron is then a polynomial of the new parameters. Changing the parameters back to the original ones gives an approximation of the Ehrhart polynomial of P[25]. Unfortunately, we do not see any simple way to obtain bounds by using this method.

6.4 Average value of the coefficients

Heine and Slovik [16] chose to take as approximation a polynomial whose coefficients equal the average value of the coefficients of the initial Ehrhart polynomial.

This method entails to compute all the elements of the periodic coefficients of the Ehrhart polynomial first: it has an exponential computation complexity, even for a fixed dimension. Moreover, it is hard to say if the obtained polynomial will vary in the same manner as the initial Ehrhart polynomial. However, the coefficient of highest degree of the approximation is exact if the coefficient is non-periodic.

6.5 Rational volume

The rational volume of P looks like a good approximation of its number of points, as it equals the rational number of unit cells in P.

To our knowledge, existing algorithms for computing such a volume first decompose P into simplices (either by triangulation or by decomposition into signed simplices) whose volumes are then summed (with signs if the simplices are signed).

The volume of a n-dimensional simplex Δ is given by

$$Vol(\Delta) = |det(v_1 - v_0, v_2 - v_0, \cdots, v_n - v_0)|/n,$$

where $v_0, v_1, v_2, \cdots, v_n$ are the n-dimensional coordinates of Δ's vertices.

This technique has been extended to the parametric case and even further to cases where inequalities involving products of parameters and variables [26].

Trying to evaluate its computation complexity leads to compare it with Barvinok's algorithm for computing Ehrhart polynomials, which also uses simplicial decomposition [4, 35], but on each of P's cones (whose vertices are parametric). Decomposing a polytope into simplices has bigger computation complexity than separately decomposing its cones. Due to the (at least) exponential complexity of these decompositions, they are likely to have a significant impact on the overall computation complexity, and therefore we speculate that it would mostly be less efficient than Barvinok's exact method.

7 Implementation and performance

The roughest approximation method, i.e. the orthogonal expansion independent of the validity domains, has been implemented in PolyLib[5] [21]. The algorithm for computing Ehrhart polynomials in PolyLib uses interpolation, counting the integer points by scanning them, for some instances of the parameters. As we reduce the period to $(1, ..., 1)^T$ we strongly reduce the number of instances needed for the interpolation. However, along with that we expand the variable space, so the number of points to scan is bigger. As a result, performance in computing approximations or bounds of Ehrhart polynomials is not always better than for computing their exact value. This is fine when approximations (or bounds) are computed for not having to handle periodic polynomials. This problem disappears when Barvinok's alogrithm, whose computational complexity does not depend on the volume of the polyhedron.

8 Perspectives

The presented approximation and bounds methods allow to give an approximate number of integer points in a parametric polyhedron. Their computational and size complexity are lower than Ehrhart polynomials, so they are relevant to software that relies on Ehrhart polynomials for *estimating* the number of integer points in a polyhedron. There already have many applications, and we believe that their importance will grow, now that their computational complexity and ease-of-use have improved. In the near future, we plan to experiment with the existing algorithms and try to integrate the expansion techniques more tightly into these.

[5]This section reflects the state of implementation at the time of the original paper (2005). In the meantime, it was implemented in the Barvinok library by S. Verdoolaege, and the expected speedup w.r.t. Polylib was confirmed.

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