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The Volume of a Lattice Polyhedron to Enumerate Processors and Parallelism

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Abstract. In the area of automatic parallelization of programs, restructuration of nested loops or parallelization of systems of affine recurrence equations use classically a geometrical model based on convex polyhedra of lattices. Since these algorithms are specified for parametric sized calculation domains, parametric lattice polyhedra have to be considered. Computing on such domains is a grand challenge in today's discrete mathematics research. In particular, the number of lattice points in a polyhedron has several useful meanings in automatic parallelization. At the end of the 50's, E. Ehrhart introduced the concept of *homothetic polyhedra* suprisingly fitting so well to our context. We recall in this paper Ehrhart's powerful results allowing to compute the number of lattice points in integral or rational polyhedra of any dimension which may be non-convex, and apply these results to the computation of the number of processors resulting from linear allocations of the calculation domain, and of the potential parallelism occurring in parallel algorithms.

1. Introduction

The type of programs to be considered in our context of automatic parallelization, is a set of nested loops, where all loop bounds are affine functions of loop counters of surrounding loops and of one size parameter, and all loop increments are equal to 1. The only dependencies are supposed to be flow-dependencies and the nested loops may not contain other control structures such as *if* statements. So, such programs can also be expressed as a system of affine recurrence equations and are modeled by

- a *convex lattice polyhedron* in $\mathbb{Z}^d$, where $\mathbb{Z}^d$ is the standard integer lattice, called the *iteration space*, with points corresponding to computed values, and d denotes the depth of the loop nest,

– vectors denoting dependencies, i.e., the parents of a point are the values used to compute this point.

Given an iteration space, a multiprocessor mapping assigns point z for processing during step $t(z)$ on a virtual processor $alloc(z)$. This mapping is subject to two constraints : a point can be computed only when its parents have been computed at previous time steps (causality). A processor cannot compute two different points during the same time step, i.e., $t(z) = t(z') \Rightarrow alloc(z) \neq alloc(z')$.

We only consider here schedules defined by an affine function $t(z) = \boldsymbol{\lambda} \cdot z + \delta$ assuming that the computation time of any point is one. We choose $\delta = -min_{z \in D}\{\boldsymbol{\lambda} \cdot z - 1\}$ in order to start the computation at time step 1.

The most classical method to allocate the points onto a virtual mesh connected array considers a linear allocation function of the form $alloc(z) = \sigma \cdot z$ where σ is a $(d-1) \times d$ integer matrix of linearly independent row-vectors. Such an allocation can be seen as a projection of the iteration space along a projection direction defined by a unimodular vector $\boldsymbol{p}$. This method respects implicitely the constraints of locality and regularity imposed by the target architecture.

The convenience of such a resulting mapping may be defined through many criteria : e.g. the potential parallelism the schedule allows, or the number of processors the allocation requires. These criteria can be defined by using nice results on polyhedra.

At the end of the 50's, E. Ehrhart introduced the concept of *homothetic polyhedra* [11], i.e. polyhedra P_n defined by a parameterized system of linear equalities and inequalities of one parameter n, such that P_n is the polyhedron resulting from the n-homothetic transformation of P_1 of center the origin O of $\mathbb{R}^d$. This model seems very opportune in our context where loop bounds are defined as affine functions of some size parameters. We recall in this paper Ehrhart's powerful results allowing to compute the number of lattice points in integral or rational polyhedra of any dimension (which may be non-convex), and apply these results to the computation of the number of processors resulting from linear allocations and of the potential parallelism occurring in parallel algorithms. It is shown that Ehrhart's method allows the determination of symbolic expressions of the number of integral points depending on the size parameter.

Many of Ehrhart's valuable results are unknown by the mathematician and computer science community at this time, in particular those concerning rational polyhedra, since many of these results were published in local reports and in french language [14]. Hence, the aim of this paper is also to finally make them known by everybody.

We have presented in previous works [7, 9, 8] another computation method of the number of integral points in integral polytopes. Since this method was closely related to the activity model of the processors, recalled in section 4, the parameterized expressions of the number of integral points are defined at time intervals, as parameterized sums of parameterized polynomials. Moreover, this method needed the parametric coordinates of all the vertices which had to be integral. Hence it yields more complex computations than the powerful mathematical results used in this paper, but it applies with any number of parameters.

2. Definitions and assumptions

We first recall some basic notions dealing with geometry of numbers [17, 23, 6] and enumerative combinatorics [25, 26]. Then some more specific concepts, closely dedicated to the scope of the paper, are introduced.

2.1. Some basic notions

A *convex polyhedron* P is a subset of $\mathbb{R}^d$ that is the intersection of a finite number of closed halfspaces. A bounded convex polyhedron is called a *convex polytope*. The *affine span* of P is the affine subspace of $\mathbb{R}^d$ entirely containing P. The *dimension*, $\dim P$, of a polyhedron P is the dimension of its affine span, and a k-dimensional polyhedron is called a k-polyhedron for brevity. The iteration spaces considered by our method are *lattice polyhedra* in the sense of the following definitions.

A lattice is the set of points $\mathbb{L}^d = \{\alpha_1 \boldsymbol{g}_1 + ... + \alpha_d \boldsymbol{g}_d \mid \alpha_1, ..., \alpha_d \in \mathbb{Z}\}$ where $\boldsymbol{g}_1, ..., \boldsymbol{g}_d$ are linearly independent vectors of $\mathbb{R}^d$, called a basis for the lattice. We denote as G be the matrix whose columns are the vectors $\boldsymbol{g}_1, ..., \boldsymbol{g}_d$. The set of all the integral points is called *standard lattice* denoted $\mathbb{Z}^d$ and any lattice $\mathbb{L}^d$ consists of the points $G \cdot z$, $z \in \mathbb{Z}^d$, i.e. $\mathbb{L}^d = G \cdot \mathbb{Z}^d$. So we get the most general lattice $\mathbb{L}^d$ by subjecting the standard lattice $\mathbb{Z}^d$ to an arbitrary non-singular transformation G [17]. Hence, without loss of generality, only the standard lattice $\mathbb{Z}^d$ is considered in the following. The enumerator of any set S is the number of points of $S \cap \mathbb{Z}^d$.

The intersection between a lattice $\mathbb{L}^d$ and the convex hull of a finite number of points $z_1, \ldots, z_{m+1}$ in $\mathbb{R}^d$ is called a *lattice polytope* P of $\mathbb{L}^d$. It is then homeomorphic to a ball $\mathbb{B}^k$. We write $k = \dim P$ and call P a k-polytope. If $z_1, \ldots, z_{m+1}$ are affinely independent points of $\mathbb{R}^d$, which means that $z_2 - z_1, z_3 - z_1, \ldots, z_{m+1} - z_1$ are linearly independent, then the convex hull of $z_1, \ldots, z_{m+1}$ is called a m-dimensional lattice simplex S of $\mathbb{L}^d$ with vertices $z_1, \ldots, z_{m+1}$. Note that the maximum number of linearly independent vectors in $\mathbb{R}^d$ is d and, hence, there could be no lattice simplex in $\mathbb{L}^d$ with more than $d+1$ vertices. A lattice polytope, or its particular case of lattice simplex, is then defined by a set of linear inequalities of the form $\{z \in \mathbb{L}^d \mid A \cdot z \leq b\}$ where A is a $m \times d$ integral matrix and b a m-dimensional integral vector.

If $\boldsymbol{c}$ is a nonzero vector and γ is defined by $\gamma = MAX\{\boldsymbol{c} \cdot z \mid A \cdot z \leq b\}$, the affine hyperplane $\{z \mid \boldsymbol{c} \cdot z = \gamma\}$ is called a supporting hyperplane of P. The *faces* of P are $\emptyset$, P, and the intersections of P with its supporting hyperplanes. Each face of P is itself a polyhedron : if P is a k-polyhedron, faces of P of dimension 0, 1 and $k-1$ are called *vertices, edges* and *facets*, respectively.

An inequality $A_i \cdot z \leq b_i$ is called an *implicit equality* in $A \cdot z \leq b$ if $A_i \cdot z = b_i$ for all z satisfying $A \cdot z \leq b$. We use the following notation:

- $A^= \cdot z \leq b^=$ is the system of implicit equalities in $A \cdot z \leq b$
- $A^+ \cdot z \leq b^+$ is the system of all other inequalities in $A \cdot z \leq b$.

There exists a one-to-one correspondence between the facets of P and the inequalities in $A^+ \cdot z \leq b^+$, given by

$$F_i = \{z \in P \mid A_i \cdot z = b_i\}$$

for any facet F_i of P and any inequality $A_i \cdot z \leq b_i$ from $A^+ \cdot z \leq b^+$. The proof can be found in [23]. It comes that any row A_i from $A^+ \cdot z \leq b^+$ defines the normal vector $\boldsymbol{A}_i$ to a facet F_i of P, and that any equation $A_i \cdot z = b_i$ from $A^+ \cdot z \leq b^+$ defines the supporting hyperplane spanning the facet F_i of P.

The partially ordered set of the faces of P, ordered by inclusion, is called the poset of P.

2.2. Some more specific notions

In the following, we consider more general lattice polyhedra which may be non-convex according to the following definition. This definition corresponds to the one given by Ehrhart in [12] and [14, p. 47] when defining *normal polyhedra.*

Definition 1. (lattice polyhedron) *A lattice polyhedron P is a closed polyhedron which admits a finite rectilinear simplicial subdivision. It is then the union of a finite number of simplices $S_1, S_2, ..., S_n$ of same dimension k such that,*

- *The intersection between two simplices either is empty or is a common face,*
- *Each facet of one of these simplices belongs to at most one other simplex, faces belonging to only one simplex being the border of the polyhedron,*
- *These simplices can be ordered such that for any $h < n$, if $P_h = S_1 \cup S_2 \cup \ldots \cup S_h$, the intersection $P_h \cap S_{h+1}$ either is empty, or is the union of common facets of S_{h+1} and the border of P_h.*

It is then homeomorphic to a ball $\mathbb{B}^k$. We write $k = dimP$ and call P a k-polyhedron.

The reader can refer to [1, 25] for investigations on combinatorial and algebraic topology.

Some results given in the following are reduced to the class of lattice polyhedra whose vertices all belong to the considered lattice. A lattice polyhedron P of $\mathbb{L}^d$ is a $\mathbb{L}^d$-polyhedron if all its vertices are in $\mathbb{L}^d$. If $\mathbb{L}^d = \mathbb{Z}^d$, P is an integral polyhedron. If some vertices of P do not belong to $\mathbb{L}^d$ and have rational coordinates, P is a rational polyhedron. Since by definition, any lattice $\mathbb{L}^d$ can be deduced from $\mathbb{Z}^d$ by a non-singular transformation G, any $\mathbb{L}^d$-polyhedron is affinely equivalent to an integral polyhedron. Hence, results given in the following and concerning integral polyhedra, can be directly extended to $\mathbb{L}^d$-polyhedra. From now on and without loss of generality, the generic term of integral polyhedra will be used.

Definition 2. (homothetic system) *A system is homothetic if its relations are of the form $\sum a_i z_i = b_i n + c_i$, $\sum a_i z_i < b_i n + c_i$, $\sum a_i z_i \leq b_i n + c_i$, where the a_i's, the b_i's and the c_i's are given integers and n a positive integral parameter. Such a system H_n is significant* [1] *if and only if H_1 defines a polyhedral domain P_1 (obviously convex). In this case, H_n defines the domain P_n, which corresponds to P_1 by the n-homothetic transformation of center O. P_1 is called the primitive domain of H_n.*

[1] The term *significant* means that all the vertices of the domain P_1 must be separated.

In the following, we consider any k-polyhedron P_n for which we know the poset and/or defined by a union of homothetic systems of the form $\{z \in \mathbb{L}^d \mid A \cdot z \leq Bn + C\}$, since any equality $\sum a_i z_i = b_i n + c_i$ is equivalent to the two inequalities $\sum -a_i z_i \leq -b_i n - c_i$ and $\sum a_i z_i \leq b_i n + c_i$.

Notation. We denote as j_n the enumerator of P_n, and i_n the enumerator of $P_n - \partial P_n$, where ∂P_n is the boundary sub-polyhedron of P_n, i.e. the set of its facets. We denote as p_n the enumerator of ∂P_n. In particular, ∂P_n is homeomorphic to the $(k-1)$-sphere $\mathbb{S}^{d-1}$.

These enumerators will be determined by *polynomials* and *pseudo-polynomials* according to the following definitions.

Definition 3. (periodic number) *A periodic number $u_n = [u_1, u_2, \ldots, u_p]$ is equal to the item whose rank is equal to n modulo p, p is called the period of u_n.*

$$u_n = \begin{cases} u_1 & if\ n = 1\ modulo\ p \\ u_2 & if\ n = 2\ modulo\ p \\ \ldots & \\ u_p & if\ n = 0\ modulo\ p \end{cases}$$

Example 1. We give some examples of periodic numbers.

- The n^{th} decimal number of the periodic decimal number $4,285\,285\ldots$ is the integral periodic number $[2, 8, 5]$.
- The general term of the serie $1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \ldots = \frac{\pi}{2\sqrt{2}}$ is $\frac{[1,1,-1,-1]}{2n-1}$.
- $(-1)^n = [-1, 1]$.
- $cos\ n\frac{2\pi}{3} = \frac{[-1,-1,2]}{2}$.
- $sin\ n\frac{\pi}{2} - cos\ n\frac{\pi}{2} = [1, 1, -1, -1]$. $\square$

Ehrhart in [14] gives the basic properties of periodic numbers.

Definition 4. (pseudo-polynomial) *A polynomial $f(n)$ is a pseudo-polynomial if some of its coefficients are periodic numbers instead of being constants. The lowest common multiple of the periods of its periodic coefficients is the pseudo-period of $f(n)$. So a pseudo-polynomial has two characteristics, its degree and its pseudo-period.*

3. The volume of a lattice polyhedron

The results given in this section are due to E. Ehrhart who made a study of the enumeration of lattice points in polyhedra, by means of polynomials in the way of Euler. All the proofs can be found in [11, 12, 13, 14]. Some aspects of Ehrhart's work were corrected, streamlined, and expanded by I. G. Macdonald [18, 19] and R. P. Stanley [25, 26].

Definition 5. (denominator) *The denominator of a rational point is the lowest common multiple of the denominators of its coordinates. The denominator of a rational polyhedron is the lowest common multiple of the denominators of its vertices.*

Ehrhart's results are based on his following fundamental theorem.

Theorem 1. (fundamental theorem) *Any enumerator of any k-polyhedron P_n is a polynomial in n of degree k if P_n is integral, and it is a pseudo-polynomial in n of degree k whose pseudo-period is the denominator of P_n if P_n is rational.*

Theorem 2. (reciprocity law) *The enumerator j_n of the k-polyhedron P_n in $\mathbb{L}^d$ is a polynomial or pseudo-polynomial $j(n)$ of degree k. The enumerator i_n of P_n is linked to j_n by the following reciprocity law,*

$$i_n = (-1)^d j(-n) \qquad j_n = (-1)^d i(-n)$$

The following results can be used for any integral lattice k-polyhedron, allowing to determine the polynomials of any enumerators from its k initial values.

Theorem 3. (Ehrhart polynomial) *For any integral k-polyhedron P_n, the enumerators verify the following identities,*

$$\{(j-j_0)^{k+1}\} = 0 \quad \{(i-i_0)^{k+1}\} = 0 \quad \{(p-p_0)^{k+1}\} = 0$$

$$j_n = \sum_{q=1}^{k} C_n^q (j-j_0)^q + j_0 \quad j^r \sim j_r,\ i^r \sim i_r,\ p^r \sim p_r$$

where $j^r \sim j_r$ means that in the expansion of any symbolic power, any power j^r must be replaced by j_r.

$$j_n = \tfrac{n(n-1)\ldots(n-k)}{k!}(j-j_0)^k \qquad i_n = \tfrac{n(n-1)\ldots(n-k)}{k!}(i-i_0)^k$$

$$p_n = \tfrac{n(n-1)\ldots(n-k+1)}{(k-1)!}(p-p_0)^{k-1} \quad j^r \sim \tfrac{j_r}{n-r},\ i^r \sim \tfrac{i_r}{n-r},\ p^r \sim \tfrac{p_r}{n-r}$$

where $j^r \sim \frac{j_r}{n-r}$ means that in the expansion of any symbolic power, any power j^r must be replaced by $\frac{j_r}{n-r}$

We give some other Ehrhart's results allowing a more direct and simplier resolution in our context.

Theorem 4. *Let P_n be a rational k-polyhedron and let the Ehrhart pseudo-polynomial be,*

$$j_n = c_k(n)n^k + c_{k-1}(n)n^{k-1} + \ldots + c_0(n)$$

where $c_0, \ldots, c_k$ are periodic numbers of n. If for some $q \in [0,k]$, the affine span of every q-dimensional face of P_1 contains a lattice point, then for any $p \geq q$, $c_p(n)$ is constant, i.e. of period one.

The fact that any face of P_1 contains a lattice point occurs mainly in the considered k-polyhedra, when they represent iteration spaces of regular nested loops with integral loop counters. The case of rational k-polyhedra whose vertices do not belong to the considered lattice, and having affine spans with no integral point, has to be frequently considered when considering cutting hyperplanes of P_n.

Due to these last theorems, the coefficients which are periodic are known, and since the period is known equal to the denominator of the k-polyhedron P_n and the degree is known equal to k, the determination of the Ehrhart pseudo-polynomial can be done by simply resolving a system of rational linear equations. Since the number of equations ne of the system must be equal to the number of unknown values, the complexity depends on the structure of P_n, its dimension, and on the size of its denominator. Moreover, ne initial values of the enumerator have to be counted.

Example 2. Consider the rational 2-polytope P_n defined by,

$$P_n = \{z \in Z^2 \mid \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \cdot z \leq \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \times n\}$$

P_1 is the closed rectangle whose vertices are $(0,0)$, $(\frac{1}{2},0)$, $(0,\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2})$. Hence their denominators are $1,2,2,2$ and the denominator of P_1 is 2. By looking at P_1, we observe that there is always an affine span of a q-dimensional face, with $q \leq 1$, containing no integer point. The only affine span containing integer points is the 2-dimensional one of the rectangle itself, containing for example point $(0,0)$. The pseudo-period is the denominator of P_1, i.e. 2. Hence, j_n is a pseudo-polynomial of degree 2 and of pseudo-period 2, the coefficients of n^1 and n^0 are periodic, and the coefficient of n^2 is constant. Hence, the pseudo-polynomial $j_n = c_1 n^2 + [c_3, c_4]n + [c_5, c_6]$ can be determined by resolving a system of 5 rational linear equations :

$$\begin{cases} c_6 & = & j_0 \\ c_1 + c_3 + c_5 & = & j_1 \\ 4c_1 + 2c_4 + c_6 & = & j_2 \\ 9c_1 + 3c_3 + c_5 & = & j_3 \\ 16c_1 + 4c_4 + c_6 & = & j_4 \end{cases}$$

From P_0 to P_4, we count $j_n = 1,1,4,4,9$. Hence, here is the Ehrhart polynomial of P_n :

$$j_n = \frac{1}{4}n^2 + [\frac{1}{2},1]n + [\frac{1}{4},1]$$

□

Computing the coefficients in the Ehrhart polynomial can be done through other approaches. For example, Barvinok in [5] proposes a complexity study of the computation of these coefficients in terms of computation of the volumes of faces, since any coefficient can be related to volumes or surface areas of P_n. But since the problem of volume computation is NP-hard [10], results given by Barvinok steps on the strong assumption of a *Volume Computation Oracle.*

Barvinok also has proposed for the first time in [4], a polynomial-time algorithm for counting integral points in non-homothetic polyhedra, i.e. when the dimension is fixed.

4. Geometrical analysis of a linear allocation

Our geometrical model of a linear multiprocessor mapping is recalled. Associated with the above results, it allows the determination of useful values in the parallelization of nested loops or systems of affine recurrence equations.

As it was said in introduction, any linear allocation of the iteration space P_n consists in a projection along a unimodular vector $\boldsymbol{p}$ called the allocation direction. Without loss of generality, the vector $\boldsymbol{p}$ is chosen such that $\boldsymbol{\lambda}\cdot\boldsymbol{p} > 0$. All the points of P_n resulting in the same virtual processor by this projection define a line-segment. Two successive points z and z' of such a line-segment, $z' - z = \boldsymbol{p}$, characterize successive computations on the virtual processor. Moreover, these two computations are time-spaced by $\boldsymbol{\lambda} \cdot \boldsymbol{p}$ steps. Hence, any virtual processor is active one step over $\boldsymbol{\lambda} \cdot \boldsymbol{p}$ steps. By considering all the so defined line-segments, their extremities define facets of P_n.

Definition 6. *Let $\boldsymbol{A}_i$ be the normal vector to any facet F_i of P_n. The inner product $\boldsymbol{A}_i \cdot \boldsymbol{p}$ determines three sets of facets of P_n :*

- *If $\boldsymbol{A}_i \cdot \boldsymbol{p} \geq 1$, F_i is a first facet denoted as F_i^+,*
- *if $\boldsymbol{A}_i \cdot \boldsymbol{p} \leq 1$, F_i is a last facet denoted as F_i^-,*
- *if $\boldsymbol{A}_i \cdot \boldsymbol{p} = 0$, F_i is a null facet denoted as F_i^0.*

So, our model only considers the first and last facets. Since only integral points are considered, the extreme integral points of the line-segments define either facets or *inside facets* of P_n according to the following definition. This case occurs for any facet such that $|\boldsymbol{A}_i \cdot \boldsymbol{p}| > 1$.

Definition 7. (inside facets) *Let F_i be a facet of P_n contained in the supporting hyperplane defined by the equation $\boldsymbol{A}_i \cdot z = b_i$. The intersection, if it is not empty, between an hyperplane of equation $\boldsymbol{A}_i \cdot z = b_i \pm \Delta$ and P_n is called an inside facet of P_n denoted as $F_{i,\Delta}$, $\Delta \in \mathbb{Z}$.*

For any first or last facet, we consider a number of inside facets equal to $|\boldsymbol{A}_i \cdot \boldsymbol{p}| - 1$, each contained in the hyperplane of equation $\boldsymbol{A}_i \cdot z = b_i \pm \Delta$, $\Delta \in [1..|\boldsymbol{A}_i \cdot \boldsymbol{p}| - 1]$. They are denoted as $F_{i,q}^+$ and $F_{i,q}^-$. From now on, but only in useful cases, F_i^+ and F_i^- will be denoted as $F_{i,0}^+$ and $F_{i,0}^-$, and in the case of common notions for $F_{i,q}^+$ and $F_{i,q}^-$, the notation $F_{i,q}^*$ will be used.

All the points of the first (or last) facets and of the first (or last) inside facets characterize the first (or last) computations of the virtual processors.

As it was said in introduction, the schedule is defined by an affine function $t(z) = \boldsymbol{\lambda} \cdot z + \delta$. The vector $\boldsymbol{\lambda}$ is the normal vector to any hyperplane defined by $H_{t_0} = \{z \in \mathbb{Z}^d \mid \boldsymbol{\lambda} \cdot z + \delta = t_0\}$, $t_0 \in \mathbb{Z}$. All the points of the iteration space P_n belonging to such an hyperplane H_{t_0} characterize simultaneous computations at step t_0.

It comes that the intersection between all the first (or last) facets, first (or last) inside facets and any hyperplane H_{t_0} characterizes all the first (or last) simultaneous computations at step t_0. The projection of this intersection along the allocation direction $\boldsymbol{p}$ characterizes the virtual processors whose first (or last) computations occur simultaneously at step t_0.

By considering the intersection between H_{t_0} and one of the first (or last) facet or first (or last) inside facet, such an intersection is either the first (or last) facet or first (or last) inside facet, or a lattice polytope with dimension less or equal than $d-2$. Since $\boldsymbol{\lambda} \cdot \boldsymbol{p} \neq 0$, the projection of this intersection along $\boldsymbol{p}$ is of the same dimension.

The projection of such an intersection $H_{t_0} \cap F_{i,q}^*$ is sensible only on a subset of virtual processors whose first (or last) computations are defined by the points of $F_{i,q}^*$. By considering the projection along $\boldsymbol{p}$ of all the couples $(H_{t_0 - \boldsymbol{\lambda} \cdot \boldsymbol{p}} \cap F_{i',q'}^-, H_{t_0} \cap F_{i,q}^+)$, we characterize three sets of virtual processors: the dead ones, the active ones and the previously unused ones, at step t_0. If any intersection $H_{t_0} \cap F_{i,q}^*$ results in $F_{i,q}^*$, its projection results in the whole subset of virtual processors for which it is defined. Otherwise, the projection of any couple $(H_{t_0-\boldsymbol{\lambda}\cdot\boldsymbol{p}} \cap F^-_{i',q'}, H_{t_0} \cap F^+_{i,q})$ results in a *diedron* according to the following definition.

Definition 8. (diedron) *Let H and H' be two half-hyperplanes of $\mathbb{Z}^d$, bounded by the same affine span D of dimension $d-2$. The couple (H, H') is called a diedron of facets H and H'. The affine span D is called the edge of the diedron.*

Any virtual processor located between the facets of the diedra at step t_0 is an active processor at t_0.

Proposition 1. (translation of the diedron) *Any diedron $(H_{t_0-\boldsymbol{\lambda}\cdot\boldsymbol{p}} \cap F^-_{i',q'}, H_{t_0} \cap F^+_{i,q})$ can be deduced from the diedron of the previous time step $(H_{t_0-\boldsymbol{\lambda}\cdot\boldsymbol{p}-1} \cap F^-_{i',q'}, H_{t_0-1} \cap F^+_{i,q})$ by a translation of the edge of the diedron along a vector $\boldsymbol{d} = \boldsymbol{D}/\boldsymbol{\lambda}\cdot\boldsymbol{D}$ where $\boldsymbol{D}$ is such that $\boldsymbol{A}_i \cdot \boldsymbol{D} = 0$ and $\boldsymbol{A}_{i'} \cdot \boldsymbol{D} = 0$.*

Proof. The edge of any diedron $(H_{t_0-\boldsymbol{\lambda}\cdot\boldsymbol{p}} \cap F^-_{i',q'}, H_{t_0} \cap F^+_{i,q})$ defines particular virtual processors whose first computation occurs at step t_0 and last computation occurs at step $t_0 - \boldsymbol{\lambda}\cdot\boldsymbol{p}$. Therefore, these virtual processors are characterized by a negative activity duration of $-\boldsymbol{\lambda}\cdot\boldsymbol{p}$ steps. It comes that the edges of two successive diedra are spaced by vectors allowing to generate the virtual processors characterized by the same activity duration. By reasoning at the level of the iteration space P_n, two processors of same activity duration are defined by two allocation line-segments containing the same number of integral points. Therefore, these two line-segments define a parallelogram. So their extreme points belonging to $F^+_{i,q}$ and their extreme points belonging to $F^-_{i',q'}$ are spaced by the same vector $\boldsymbol{D}$, i.e. $\boldsymbol{A}_i \cdot \boldsymbol{D} = 0$ and $\boldsymbol{A}_{i'} \cdot \boldsymbol{D} = 0$. Moreover, in order to translate by a one time step, this vector is normalized. □

This linear multiprocessor schedule modeling allows the determination of several useful values by means of combinatorial computations on lattice polyhedra. In the next section, a direct application of Ehrhart's method is developped.

5. Computation of indicators

Now these great results are used for the computation of

- the processor count in the virtual grid allocated from any linear allocation,
- the potential parallelism associated with any linear time schedule.

In fact, any indicator resulting from lattice points enumeration in lattice polyhedra could have been considered. We just limit ourselves to these two items which seem to be representative of this kind of computations.

- *Processor count*

Our geometrical model of a linear allocation presented above characterizes the first (or last) computations of all the virtual processors as being points of first (or last) facets and first (or last) inside facets. Since it is obvious that any virtual processor has only one first (or last) computation, the processor count is given by the number of integral points in all the first (or last) facets and first (or last) inside facets.

Proposition 2. (processor count) *Let $|P|$ be the number of integral points in P. The processor count p_n resulting from any linear allocation of P_n is given by,*

$$p_n = |F^*| = |\cup_{i=1}^{f} F_i^*| = |\cup_{i=1}^{f} \cup_{q=0}^{|\boldsymbol{A}_i \cdot \boldsymbol{p}|-1} F_{i,q}^*|$$

where f denotes the number of first (or last) facets.

Any facet or inside facet $F_{i,q}^*$ is a $(k-1)$-dimensional polytope since P_n is a k-polytope. Moreover, it is a rational polytope : its vertices have rational coordinates since any $F_{i,q}^*$ can be seen as the intersection between an hyperplane and P_n. Hence, for any fixed i, the set F_i^* is a k-dimensional rational polytope defined by

$$F_i^* = \{z \in \mathbb{Z}^d \mid A \cdot z \le Bn + C, A_i \cdot z \ge b_i n + c_i \pm (|A_i \cdot \boldsymbol{p}| - 1)\}$$

In the general case $f > 1$, the union of all these rational polytopes is a non-convex rational polyhedron and Ehrhart's results can apply. Such a polyhedron belongs to the set of *normal polyhedra* according to the definition of Ehrhart in [12, 14] and to the one given in section 2.

The determination of the Ehrhart polynomial for such a rational polyhedron yields first the computation of the rational coordinates of its vertices. Finding all the vertices of a convex polyhedron is a well-known problem. A survey and comparison of methods can be found in [20]. For example, Chernikova's algorithm [20] or more recently Avis and Fukuda's pivoting algorithm [2] can be used. But since our considered polyhedron F^* is non-convex in a general way, the following heuristic procedure is needed to compute all the vertices of F^* for $n = 1$:

- Compute all the vertices of each rational polytope F_i^*, for $n = 1$ and $i \in [1..f]$.
- For each found vertex v, v is a vertex of F^* if and only if it is a vertex of a polytope F_i^* and it does not belong to any other F_j^*, unless it is also a vertex of F_j^*.
- Compute all the vertices of each intersection $F_i^* \cap F_j^*$, for $n = 1$, $i \neq j$ and $i, j \in [1..f]$.
- For each found vertex v, v is a vertex of F^* if and only if it is not a vertex of any F_i^*.

Example 3. A simple 3-dimensional example is shown on figure 1 :

- v_1 is a vertex of F_1^* and does not belong to F_2^*. Hence, v_1 is a vertex of F^*.
- v_2 is a vertex of F_1^* and is also a vertex of F_2^*. Hence, v_2 is a vertex of F^*.
- v_3 is a vertex of F_1^* and belongs to F_2^* without being one of its vertices. Hence, v_3 is not a vertex of F^*.
- v_4 is a vertex of $F_1^* \cap F_2^*$ and is not a vertex, neither of F_1^* nor of F_2^*. Hence, v_4 is a vertex of F^*. □

The Ehrhart polynomial of F^*, giving the processor count in the grid allocated from any linear allocation, is then determined by using directly the results presented in section 3.

• *Potential parallelism*

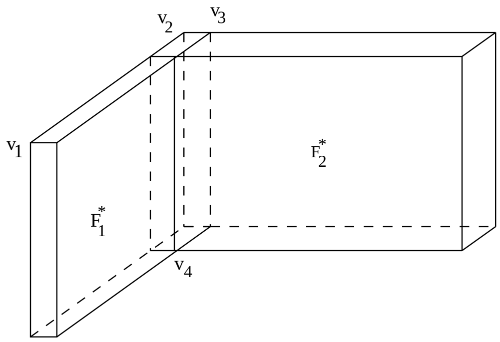

Fig. 1. A 3D illustration for the vertex finding procedure of non-convex sets F^*

Through our geometrical model, the potential parallelism of any time step t_0, i.e. the number of simultaneous computations at t_0, is equal to the number of integral points in $P_{n,t_0} = H_{t_0} \cap P_n$, i.e. $pp(t_0) = |P_{n,t_0}|$. It is defined by

$$P_{n,t_0} = \{z \in \mathbb{Z}^d \mid A \cdot z \leq Bn + C, \boldsymbol{\lambda} \cdot z + \delta = t_0\}$$

This set P_{n,t_0} is a $(k-1)$-dimensional rational polytope. Moreover, it is a parametric polytope with two parameters n and t_0. But since Ehrhart's method applies only with one parameter, we choose to express t_0 as a polynomial in n of degree 1, i.e. $t_0 = xn + y$, where x and y are rational.

$$P_{n,xn+y} = \{z \in \mathbb{Z}^d \mid A \cdot z \leq Bn + C, \boldsymbol{\lambda} \cdot z + \delta = xn + y\}$$

By computing the vertices of $P_{1,x+y}$, the Ehrhart polynomial, giving the potential parallelism $pp(t_0)$ of any step $t_0 = xn + y$, is directly determined. An opportune choice of the coefficients x and y is made by computing the latency τ of the parallel algorithm, i.e. $\tau = MAX_{z \in P_n} \boldsymbol{\lambda} \cdot z + \delta$. At least one vertex of P_n, say v, is such that $t(v) = \tau$. The parametric coordinates of this vertex, depending on n, allows to get a polynomial expression of τ in n of degree 1. Hence, we let the coefficient of n of this expression be the numerator of x, and the constant term be the numerator of y. For example, with such coefficients x and y, the potential parallelism occurring at mid-time can be expressed as a pseudo-polynomial in n.

Moreover, such a parametric expression of t_0 allows to find the step where $pp(t_0)$ is maximum by enumeration of some initial values and by the knowledge of the following theorem found in [15, p. 87], since the potential parallelism associated to a given linear time-schedule, i.e. the maximum number of simultaneous computations, is defined by $pp = MAX_{t_0}\ pp(t_0)$.

Theorem 5. (Brunn's theorem) *The volume variation of parallel hyperplane sections of a convex set occurs in at most three phases : increasing, constant, decreasing.*

The whole method is now applied to the Gauss elimination algorithm.

6. Example : The Gaussian elimination algorithm

This well-known algorithm can be expressed as the following nested loops :

FOR $k := 1$ TO $N-1$ DO
    FOR $i := k+1$ TO N DO
        FOR $j := k+1$ TO $N+1$ DO
            $a[i,j,k] := a[i,j,k-1] - a[i,k,k-1] * a[k,j,k-1]/a[k,k,k-1]$
        ENDFOR
    ENDFOR
ENDFOR

The associated iteration space P_N is defined by

$$P_N = \{z \in \mathbb{Z}^3 \mid \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot z \leq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} N + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \}$$

Since P_0 and P_1 are empty, and P_2 does not show all the vertices, this domain is significant from $N = 3$. Hence, in order to define the iteration space as an homothetic polyhedron P_n, we let $N = n+3$ and define it as follows. It is represented on figure 2 where the parametric coordinates of the vertices depending on n are given.

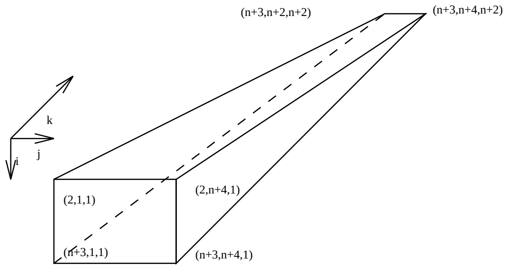

Fig. 2. Gauss iteration space

$$\begin{aligned} P_n &= \{z \in \mathbb{Z}^3 \mid A \cdot z \leq Bn + C\} \\ &= \{z \in \mathbb{Z}^3 \mid \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot z \leq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} n + \begin{pmatrix} -1 \\ 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} \} \end{aligned}$$

Let us consider the linear allocation defined by the projection vector $\boldsymbol{p} = (1,-1,1)$ and the time schedule defined by $t(z) = (1,1,1) \cdot z - 3$. Such a choice leads to the following parallel program obtained through the *Fourier-Motzkin*

Elimination Method [3], where the first coordinate denotes the calculation step and the second and third coordinates denote the processor where the item is computed.

FOR $t := 1$ TO $3N-3$ DO
  FORALL $(x = MAX(t-N+4, \frac{2t+7}{3}) : min(2N+1, t+2))$
  FORALL $(y = MAX(t-x-N+3, 2t-3x+6) : min(t-2x+N+4, -1))$
    $a[t+3, x, y] := a[t+2, x, y-1]$
      $-a[3t-3x-y+8, 2t-2x-y+6, y-1] * a[t+y+2, x+y, -1]$
      $/a[3t-3x+8, 2t-2x+6, -1]$
ENDFOR

In order to implement such a program, many related informations can be useful. In particular, the number of processors (x, y) used by the program and the maximum parallelism, i.e. the maximum number of processors (x, y) simultaneously active at a given step t, play an important role.

The normal vector to any facet F_i of P_n is given by any row of A. But in order to classify the facets in first, last and null facets, the normal vectors have to be directed to the inside of P_n. We get the following normal vectors :

$$\boldsymbol{A}_1 = (1, 0, -1) \quad \boldsymbol{A}_2 = (-1, 0, 0) \quad \boldsymbol{A}_3 = (0, 1, -1) \quad \boldsymbol{A}_4 = (0, -1, 0) \quad \boldsymbol{A}_5 = (0, 0, 1)$$

The inner product $\boldsymbol{A}_i \cdot \boldsymbol{p}$ indicates that

- F_1 is a null facet F_1^0.
- F_2 and F_3 are last facets F_2^- and F_3^-. Moreover, F_3^- is associated to one inside facet $F_{3,1}^-$. Hence, we consider F_2^-, $F_{3,0}^-$ and $F_{3,1}^-$ represented on figure 3.
- F_4 and F_5 are first facets F_4^+ and F_5^+ represented on figure 3.

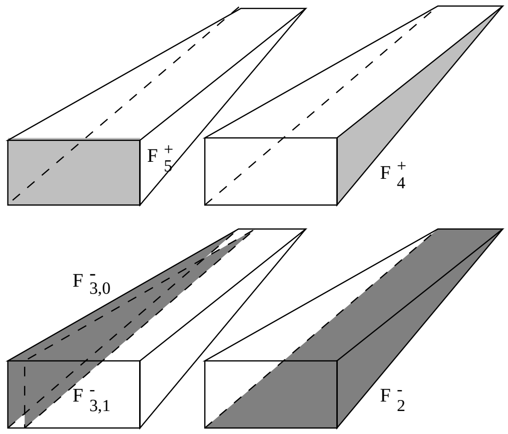

Fig. 3. First and last facets defined by $\boldsymbol{p}$

- *Processor count*

We determine the processor count resulting from the projection along $\boldsymbol{p}$ by considering the first facets. The set $F_4^+ \cup F_5^+$ is a normal integral polyhedron, union of two homothetic 2-polyhedra. Theorem 3 can be applied :

$$\begin{aligned} j_n &= \sum_{q=1}^{2} C_n^q (j-j_0)^q + j_0 \\ &= n(j_1 - j_0) + \tfrac{n(n-1)}{2}(j_2 - 2j_1 + j_0) + j_0 \end{aligned}$$

By counting the number of points in $F_4^+ \cup F_5^+$ for $n = 0$, $n = 1$ and $n = 2$, we get $j_0 = 9$, $j_1 = 18$ and $j_2 = 30$. Hence, the processor count is equal to :

$$p_n = \frac{3(n^2+5n+6)}{2}, \ i.e. \ p_N = \frac{3N(N-1)}{2}$$

The same result can be determined by considering the last facets.

- *Potential parallelism*

The latency of the parallel algorithm is given by $\tau = MAX_{z \in P_n}(1,1,1) \cdot z - 3 = 3n+6$. The potential parallelism occurring at mid-time is defined by $P_{n,\frac{3n+6}{2}}$ if $n \ mod \ 2 = 0$ and by $P_{n,\frac{3n+7}{2}}$ if $n \ mod \ 2 = 1$.

- Case $n \ mod \ 2 = 0$

The set $P_{n,\frac{3n+6}{2}}$ is defined by

$$P_{n,\frac{3n+6}{2}} = \{z \in \mathbb{Z}^3 \mid A \cdot z \le Bn + C, (1,1,1) \cdot z - 3 = \frac{3n+6}{2}\}$$

Let $n = 2m$ and define $P_{n,\frac{3n+6}{2}}$ as :

$$P_{m,3m+3} = \{z \in \mathbb{Z}^3 \mid A \cdot z \le 2m \times B + C, (1,1,1) \cdot z - 3 = 3m+3\}$$

This set is a rational 2-polytope. The vertices of $P_{1,6}$ are computed and represented on figure 4 :

$$(2,6,1) \quad (5,3,1) \quad (5,2,2) \quad (\tfrac{11}{3}, \tfrac{8}{3}, \tfrac{8}{3})$$

The denominator of $P_{1,6}$ is equal to the denominator of the rational vertex, i.e. 3. Hence, the Ehrhart polynomial is of period 3 and of degree 2. By applying theorem 4, we observe that there is one affine span of a 0-dimensional face, i.e. a vertex, which is not an integral point. But for any $q > 0$, the affine span of any q-dimensional face contains integral points. Hence, the coefficients of m^2 and m^1 are constant. It comes that the five coefficients c_1, c_2, c_3, c_4, c_5 of the pseudo-polynomial $c_1 m^2 + c_2 m + [c_3, c_4, c_5]$ have to be determined by resolving the following linear system of 5 equations :

$$\left\{ \begin{array}{lcl} c_5 & = & j_0 \\ c_1 + c_2 + c_3 & = & j_1 \\ 4c_1 + 2c_2 + c_4 & = & j_2 \\ 9c_1 + 3c_2 + c_5 & = & j_3 \\ 16c_1 + 4c_2 + c_3 & = & j_4 \end{array} \right.$$

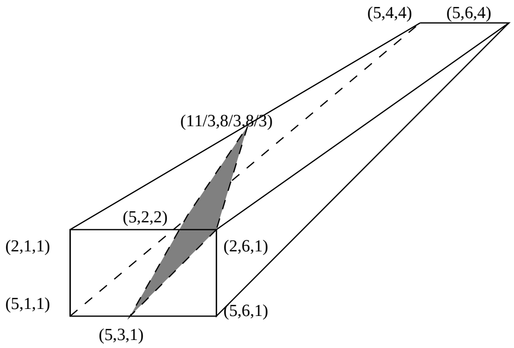

Fig. 4. $P_{1,6}$ and its vertices

By counting the number of points in $P_{0,3}$, $P_{1,6}$, $P_{2,9}$, $P_{3,12}$ and $P_{4,15}$, we get $j_0 = 2$, $j_1 = 6$, $j_2 = 12$, $j_3 = 20$ and $j_4 = 30$. Hence, the resolution of the above system gives the following results :

$$c_1 = 1 \quad c_2 = 3 \quad c_3 = 2 \quad c_4 = 2 \quad c_5 = 2$$

Hence, the potential parallelism occurring at mid-time when $n \ mod \ 2 = 0$ is a polynomial :

$$pp(3m+3) = m^2 + 3m + [2,2,2] = m^2 + 3m + 2$$

$$pp(\tfrac{3n+6}{2}) = \tfrac{1}{4}n^2 + \tfrac{3}{2}n + 2$$

$$pp(\tfrac{3N-3}{2}) = \tfrac{1}{4}N^2 - \tfrac{1}{4}$$

- Case $n \ mod \ 2 = 1$

The set $P_{n,\frac{3n+7}{2}}$ is defined by

$$P_{n,\frac{3n+7}{2}} = \{z \in \mathbb{Z}^3 \mid A \cdot z \leq Bn + C, (1,1,1) \cdot z - 3 = \frac{3n+7}{2}\}$$

Let $n = 2m+1$ and define $P_{n,\frac{3n+7}{2}}$ as :

$$P_{m,3m+5} = \{z \in \mathbb{Z}^3 \mid A \cdot z \leq (2m+1) \times B + C, (1,1,1) \cdot z - 3 = 3m+5\}$$

This set is a rational 2-polytope. The vertices of $P_{1,8}$ are computed.

$$(\tfrac{5}{2}, 7, \tfrac{3}{2}) \quad (6,4,1) \quad (6, \tfrac{5}{2}, \tfrac{5}{2}) \quad (\tfrac{13}{3}, \tfrac{10}{3}, \tfrac{10}{3})$$

The denominator of $P_{1,8}$ is equal to the lowest common multiple of the denominators of the rational vertices, i.e. 6. Hence, the Ehrhart polynomial is of period 6 and of degree 2. By applying theorem 4, we observe that there is at least one affine span of a 1-dimensional face, i.e. the edge containing vertices $(\frac{5}{2}, 7, \frac{3}{2})$ and $(\frac{13}{3}, \frac{10}{3}, \frac{10}{3})$, which contains no integral point. Hence, only the coefficient of

m^2 is constant. It comes that the thirteen coefficients of the pseudo-polynomial $c_1 m^2 + [c_2, c_3, c_4, c_5, c_6, c_7]m + [c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}]$ have to be determined by resolving the following linear system of 13 equations :

$$\left\{ \begin{array}{lcl} c_{13} & = & j_0 \\ c_1 + c_2 + c_8 & = & j_1 \\ 4c_1 + 2c_3 + c_9 & = & j_2 \\ 9c_1 + 3c_4 + c_{10} & = & j_3 \\ 16c_1 + 4c_5 + c_{11} & = & j_4 \\ 25c_1 + 5c_6 + c_{12} & = & j_5 \\ 36c_1 + 6c_7 + c_{13} & = & j_6 \\ 49c_1 + 7c_2 + c_8 & = & j_7 \\ 64c_1 + 8c_3 + c_9 & = & j_8 \\ 81c_1 + 9c_4 + c_{10} & = & j_9 \\ 100c_1 + 10c_5 + c_{11} & = & j_{10} \\ 121c_1 + 11c_6 + c_{12} & = & j_{11} \\ 144c_1 + 12c_7 + c_{13} & = & j_{12} \end{array} \right.$$

By counting the number of points in $P_{0,5}$, $P_{1,8}$, $P_{2,11}$, $P_{3,14}$, $P_{4,17}$, $P_{5,20}$, $P_{6,23}$, $P_{7,26}$, $P_{8,29}$, $P_{9,32}$, $P_{10,35}$, $P_{11,38}$ and $P_{12,41}$, we get $j_0 = 4$, $j_1 = 9$, $j_2 = 16$, $j_3 = 25$, $j_4 = 36$, $j_5 = 49$, $j_6 = 64$, $j_7 = 81$, $j_8 = 100$, $j_9 = 121$, $j_{10} = 144$, $j_{11} = 169$ and $j_{12} = 196$. Hence, the resolution of the above system gives the following results :

$$c_1 = 1 \quad c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = c_{12} = c_{13} = 4$$

Hence, the potential parallelism occurring at mid-time when $n \ mod \ 2 = 1$ is a polynomial.

$$pp(3m+5) = m^2 + [4,4,4,4,4,4]m + [4,4,4,4,4,4] = (m+2)^2$$

$$pp(\tfrac{3n+7}{2}) = \left(\tfrac{n+3}{2}\right)^2$$

$$pp(\tfrac{3N-2}{2}) = \left(\tfrac{N}{2}\right)^2$$

The above results can be summerized by the following pseudo-polynomial since the potential parallelism at mid-time consists in considering step $\lceil \frac{\tau}{2} \rceil$ for any value of n :

$$pp(\lceil \frac{\tau}{2} \rceil) = \frac{1}{4}n^2 + \frac{3}{2}n + [\frac{9}{4}, 2] = \frac{1}{4}N^2 + [-\frac{1}{4}, 0]$$

In order to determine the potential parallelism pp of the time-schedule by using Brunn's theorem, we determine in the same way some other polynomials :

- $n \ mod \ 2 = 0$

$$\begin{array}{l} pp(3m+2) = m^2 + 3m + 1 \\ pp(3m+3) = m^2 + 3m + 2 \\ pp(3m+4) = m^2 + 3m + 2 \\ pp(3m+5) = m^2 + 3m + 2 \\ pp(3m+6) = m^2 + 3m + 1 \end{array}$$

These results show clearly that when $n \ mod \ 2 = 0$, the potential parallelism increases until time-step $3m+3$, keeps constant from time-step $3m+3$ to time-step

$3m+5$, and then decreases. Hence, according to Brunn's theorem, the potential parallelism of time-step $3m+3$ previously determined is maximum and is equal to the potential parallelism pp of the time-schedule.

- $n \ mod \ 2 = 1$

$$\begin{aligned} pp(3m+4) &= m^2+4m+3 \\ pp(3m+5) &= m^2+4m+4 \\ pp(3m+6) &= m^2+4m+4 \\ pp(3m+7) &= m^2+4m+3 \end{aligned}$$

These results show clearly that when $n \ mod \ 2 = 1$, the potential parallelism increases until time-step $3m+5$, keeps constant from time-step $3m+5$ to time-step $3m+6$, and then decreases. Hence, according to Brunn's theorem, the potential parallelism of time-step $3m+5$ previously determined is maximum and is equal to the potential parallelism pp of the time-schedule. So, we have :

$$\begin{aligned} pp = pp(\lceil \tfrac{7}{2} \rceil) &= \tfrac{1}{4}n^2 + \tfrac{3}{2}n + [\tfrac{9}{4}, 2] \\ &= \tfrac{1}{4}N^2 + [-\tfrac{1}{4}, 0] \end{aligned}$$

7. Conclusion

It has been shown that Ehrhart's powerful results allow to resolve efficiently the problem of counting the number of integral points in integral and rational homothetic polyhedra, in their linear projections, and in their intersections with cutting hyperplanes. In our context of automatic parallelization of programs, these results yields an easy determination of the number of processors resulting from linear allocations and of the potential parallelism resulting from affine time-schedules. These two quantities are representative examples for such kind of computations. But the same method can be used to compute, for example, the number of calculations (or processors) using a same data.

But it is not generalized at this time to parametric polyhedra of any number of size parameters. Ehrhart in [14] started a limited study of bipartite systems with two parameters. He ends his study with the following conjecture.

Conjecture 1. For any significant diophantine linear system of any dimension, linearly dependent of several positive integral parameters, the enumerator depending on these parameters is expressed at different intervals by different pseudo-polynomials.

To our knowledge, it is still an opened problem at this moment. But it gets more and more necessary to have new results in this research area, since the concept of parametric domains constitutes one of the major foundations in the theory of automatic parallelization of programs [16, 22]. In particular, it seems to be opportune to introduce the concept of multi-dimensional homothetic systems and multi-dimensional periodic numbers : this is the subject of our future works. Nevertheless, recall that our own method, previously presented in [7, 9, 8], also applied on the Gaussian elimination, can apply for any number of size parameters, but needs the parametric coordinates of the vertices and is limited to integral polyhedra. Hence, it yields more complex computations.

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