Counting Solutions to Linear and Nonlinear Constraints through Ehrhart polynomials:

Applications to Analysis and Transformation of Scientific Programs

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Objective of the method

To provide a general computation module to compilers for the **analysis and optimization of nested loops**.

Analysis involving **parametric enumerations** of the number of integer solutions of a set of linear constraints:

... *number of flops, number of memory locations touched, number of cache lines touched, number of array elements to be transmitted,* ...

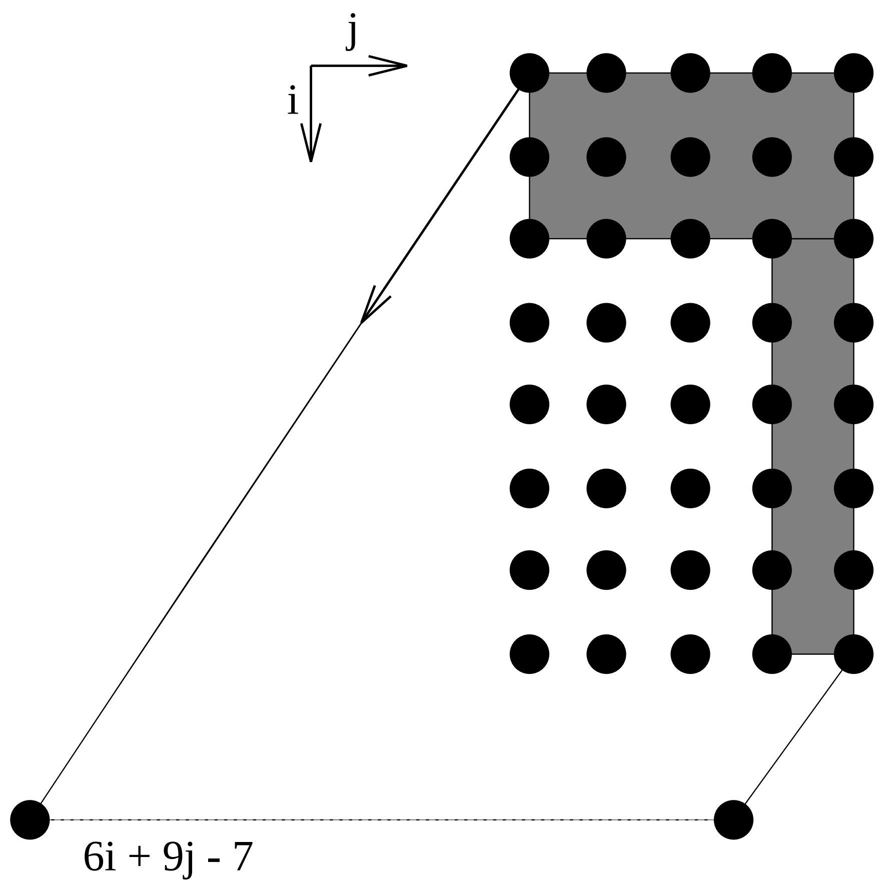
... *load balancing, message traffic evaluation, message buffers allocation,* ...

Objective of the method

For example . . .

Number of distinct memory locations
touched by
(cache effectiveness evaluation)

```
for i:=1 to n+3
    for j:=1 to n
        a(6i+9j-7) = a(6i+9j-7) + 5
```



Mathematical background

Results due to the french mathematician
Eugène Ehrhart
from the end of the 50's.

Expanded and proved by
I.G. Macdonald and R.P. Stanley.

Enumeration of lattice points in
polytopes depending on
one positive integer parameter,
by means of polynomials.

Expanded to
any number of positive integer parameters.

Steps of the method

1. Model the problem as a counting of integer points in a **parametric convex polytope** P or in a **union of parametric convex polytopes** P
   ($\Rightarrow$ Ehrhart's *normal polyhedra*).

2. Compute the **parametric coordinates of the vertices** of P and **the domains where they are constant**.

3. For each domain, count the number of points of P for **some initial low values of the parameters**.

4. For each domain, solve a **system of linear equations** giving the **Ehrhart polynomials** of P.

Ehrhart polynomials

Parameters $n_1, n_2, \ldots, n_p$

Ehrhart polynomial of degree k
= **multi-variate pseudo-polynomial**:

$$\sum_{i_1=0}^{k} \sum_{i_2=0}^{k} \cdots \sum_{i_p=0}^{k} c_{i_1,i_2,\ldots,i_p} \, n_1^{i_1} n_2^{i_2} \ldots n_p^{i_p}$$

$c_{i_1,i_2,\ldots,i_p}$ = **periodic coefficients**:

$\Downarrow$

Array of dimension at most p

$\Downarrow$

Giving the value of $c_{i_1,i_2,\ldots,i_p}$ indexed by the **parameters modulo the number of possible values**
= periods of the periodic coefficients

Least common multiple of the periods
= **pseudo-period** of the pseudo-polynomial

Examples of Ehrhart polynomials

$$f(n) = \frac{1}{8}n^2 + \left[\frac{1}{2}, \frac{3}{4}\right] n + \left[\frac{3}{8}, 1\right]$$

$$\Leftrightarrow$$

$$f(n) = \begin{cases} \frac{1}{8}n^2 + \frac{1}{2}n + \frac{3}{8} & if \;\; n \;\; mod \; 2 = 1 \\ \frac{1}{8}n^2 + \frac{3}{4}n + 1 & if \;\; n \;\; mod \; 2 = 0 \end{cases}$$

Examples of Ehrhart polynomials

$$f(n,m) =$$

$$\frac{1}{4}nm + \left[\frac{1}{4}, \frac{1}{2}\right]_m n + \left[\frac{1}{4}, \frac{1}{2}\right]_n m + \left[\begin{array}{cc} 1/4 & 1/2 \\ 1/2 & 1 \end{array}\right]_{n,m}$$

$$\Leftrightarrow$$

$$f(n,m) = \begin{cases} \frac{1}{4}nm + \frac{1}{4}n + \frac{1}{4}m + \frac{1}{4} \\ if\ n\ mod\ 2 = 1\ and\ m\ mod\ 2 = 1 \\ \\ \frac{1}{4}nm + \frac{1}{2}n + \frac{1}{4}m + \frac{1}{2} \\ if\ n\ mod\ 2 = 1\ and\ m\ mod\ 2 = 0 \\ \\ \frac{1}{4}nm + \frac{1}{4}n + \frac{1}{2}m + \frac{1}{2} \\ if\ n\ mod\ 2 = 0\ and\ m\ mod\ 2 = 1 \\ \\ \frac{1}{4}nm + \frac{1}{2}n + \frac{1}{2}m + 1 \\ if\ n\ mod\ 2 = 0\ and\ m\ mod\ 2 = 0 \end{cases}$$

Ehrhart's fundamental results

Some definitions . . .

INTEGRAL POLYTOPE

= All the vertices have **integral coordinates**

RATIONAL POLYTOPE

= At least one vertex has **rational coordinates**

DIMENSION OF A POLYTOPE

= Dimension of the smallest affine subspace entirely containing the polytope

DENOMINATOR OF A POLYTOPE

= Least common multiple of the denominators of the vertices coordinates

Ehrhart's fundamental results

Some definitions ...

HOMOTHETIC POLYTOPE P_n

$$\sum a_i z_i = b_n$$

$$\sum a_i z_i < b_n$$

$$\sum a_i z_i \leq b_n$$

a_i, b given integers
x_i free variable
n positive integral parameter

P_n results from the n-**homothetic transformation of** P_1

P_n obviously convex

Ehrhart's fundamental results

Some definitions ...

BORDERED POLYTOPE P'_n

$$\sum a_i z_i = b_n + c$$

$$\sum a_i z_i < b_n + c$$

$$\sum a_i z_i \leq b_n + c$$

a_i, b given integers
c rational
x_i free variable
n positive integral parameter

By deleting all the c's
$\Rightarrow$ Homothetic polytope P_n = **kernel of** P'_n

P'_n obviously convex

Ehrhart's fundamental results

The fundamental theorem ...

Parametric number of integer points
of **any homothetic or bordered polytope**
whose dimension is k

$$\Downarrow$$

IF P_n IS INTEGRAL

= A POLYNOMIAL IN n
OF DEGREE k

$$\Downarrow$$

IF P_n IS RATIONAL

= A PSEUDO-POLYNOMIAL IN n
OF DEGREE k
WHOSE PSEUDO-PERIOD IS THE
DENOMINATOR OF P_n

Ehrhart's fundamental results

Application ...

DEGREE k of the Ehrhart polynomial
= Dimension of P_n

PSEUDO-PERIOD d
of the Ehrhart polynomial
= Denominator of P_n

($\Rightarrow$ Computation of the parametric
vertices coordinates of P_n)
(Loechner & Wilde algorithm)

Computation of the exact formulae :

$\rightarrow$ Counting of some initial values
$(k+1) \times d$ unknowns
($\Rightarrow$ Scanning loop / Fourier-Motzkin)

$\rightarrow$ Solving a system of linear equations

Ehrhart's fundamental results

Application ...

Minimizing the number of unknowns

$\rightarrow$ Theorem :

$$f(n) = c_k n^k + c_{k-1} n^{k-1} + \ldots + c_0$$

If for some $q \in [0, k]$,

any affine subspace containing a q-dimensional face contains at least one integer point

then $\forall\ p \geq q$,

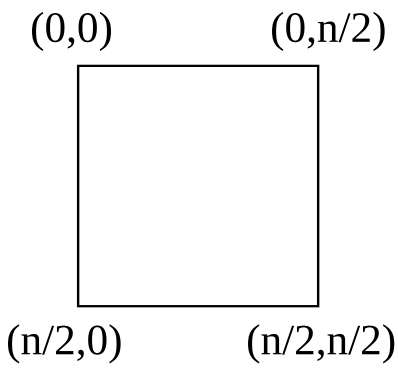
c_p **is constant (of period one)**

Ehrhart's fundamental results

a simple example ...

The parametric square defined by

$$0 \le i \le \frac{n}{2} \qquad\qquad 0 \le j \le \frac{n}{2}$$



$\Rightarrow$ Ehrhart polynomial of the form :

$$f(n) = c_1 n^2 + [c_2, c_3]n + [c_4, c_5]$$

$\Rightarrow$ System of 5 linear equations :

$$\begin{cases} c_5 & = f(0) = 1 \\ c_1 + c_2 + c_4 & = f(1) = 1 \\ 4c_1 + 2c_3 + c_5 & = f(2) = 4 \\ 9c_1 + 3c_2 + c_4 & = f(3) = 4 \\ 16c_1 + 4c_3 + c_5 & = f(4) = 9 \end{cases}$$

$\Rightarrow f(n) = \frac{1}{4}n^2 + [\frac{1}{2}, 1]n + [\frac{1}{4}, 1]$

An application example

Analysis of a parallel loop ...

```
for t:=1 to 2*n+1
    forall x:=0 to (t-1)/2
        forall y:=max(t-x-n-1,0) to (t-2*x-1)/2
            Statement(s)
        endforall
    endforall
endfor
```

$\rightarrow$ Number of virtual processors (x, y)
allocated $= np$?

$\rightarrow$ Maximum number of simultaneously
active processors $= maxpar$?

$\Rightarrow$ **Efficiency of the virtual processor grid**

$= maxpar/np$ **?**

An application example

Analysis of a parallel loop . . .

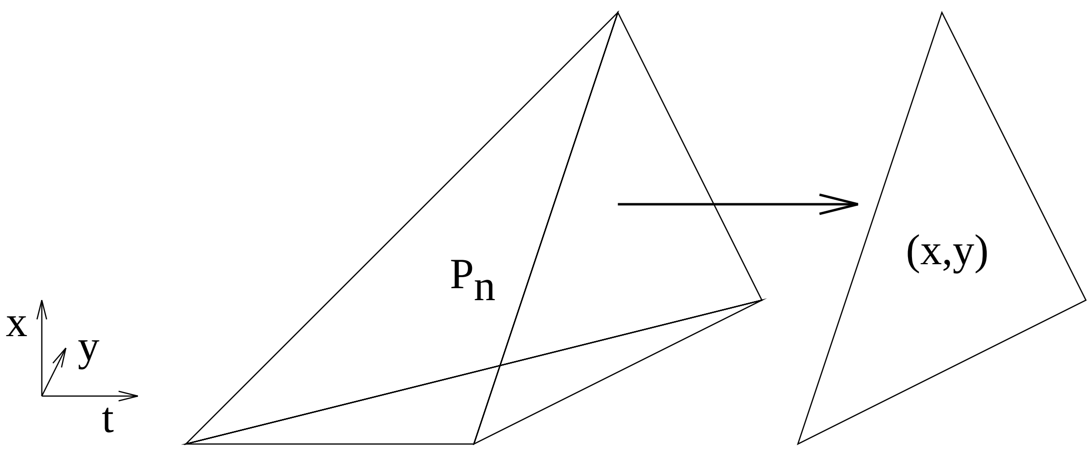
$$np$$
$$=$$
$$\text{number of distinct values of } (x, y)$$

$$P_n = (t, x, y) \in Z^3 \text{ such that}$$

$$1 \le t \le 2n + 1 \qquad 0 \le x \le \frac{t-1}{2}$$

$$0 \le y \le \frac{t-2x-1}{2} \qquad t - x - n - 1 \le y$$

$\Rightarrow np$ **= number of integer points in the projection of** P_n **along** t



An application example

Analysis of a parallel loop ...

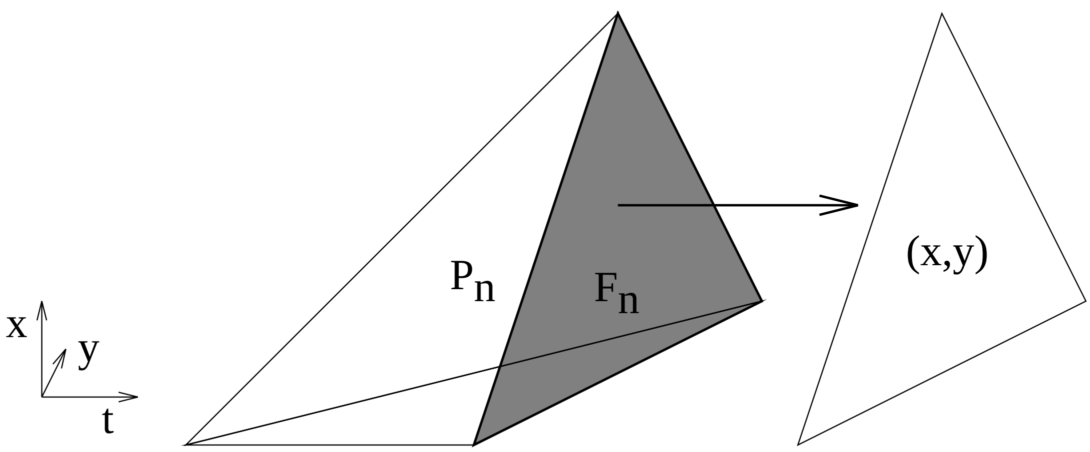
Any (x, y) results from a line segment of points of P_n parallel to the t axis

Extremities of all these line segments describe facets of P_n

Number of points of one side extremities (first or last)

=

Number of points in the projection of P_n



$\Rightarrow$ One facet F_n defined by

$$1 \le t \le 2n+1 \qquad 0 \le x \le \frac{t-1}{2}$$

$$0 \le y \le \frac{t-2x-1}{2} \qquad t - x - n - 1 = y$$

An application example

Analysis of a parallel loop ...

Computation of
the Ehrhart polynomial of F_n

$\rightarrow$ Vertices of F_n
(Loechner & Wilde algorithm) :

$$(n+1,0,0) \quad (2n+1,n,0) \quad (2n+1,0,n)$$

$\Rightarrow$ Ehrhart polynomial of the form:
$$f(n) = c_1 n^2 + c_2 n + c_3$$

$\rightarrow$ Computation of 3 initial values
(scanning loop: Fourier-Motzkin)

```
init_val:=0;
for t:=n+1 to 2*n+1
    for x:=0 to -n+t-1
        for y:=-n+t-x-1 to -n+t-x-1
            init_val:=init_val+1;
```

An application example

Analysis of a parallel loop ...

A Maple package on Ehrhart polynomials ...

```
    |\^/|     Maple V Release 2 (Universite Louis Pasteur)
._|\|   |/|_. Copyright (c) 1981-1993 by the University of Waterloo.
 \  MAPLE  /  All rights reserved. Maple and Maple V are registered
 <____ ____>  trademarks of Waterloo Maple Software.
      |       Type ? for help.
> read(`Ehrhart.m`);
> init:=array([1,3,6],0..2);

                  init := array(0 .. 2,, [
                              0 = 1
                              1 = 3
                              2 = 6
                          ])

> binf:=array([0],1..1);

                       binf := [ 0 ]

> Ehrhartpoly(Recur(Pdt(1,1,1)),init,binf,3,1,j);

                           2
                   1/2 n1   + 3/2 n1 + 1
```

⇒ **Number of allocated processors**

$$np = \frac{1}{2}n^2 + \frac{3}{2}n + 1$$

An application example

Analysis of a parallel loop ...

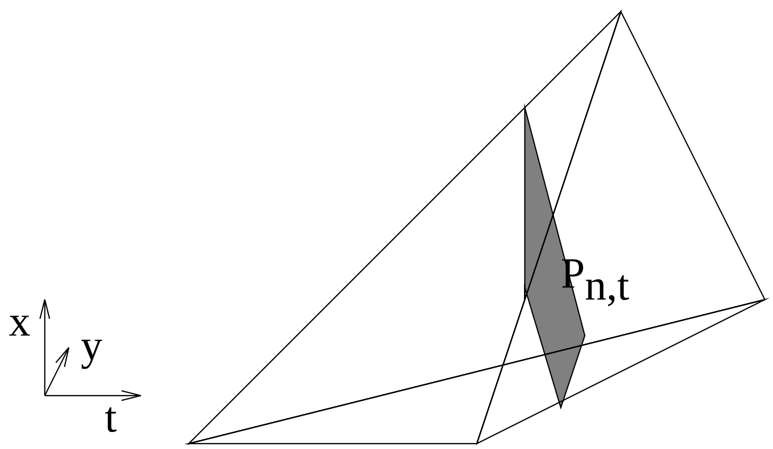
$maxpar$ = maximum number of active processors (x, y) for $t \in [1..2n+1]$

$$P_{n,t} = (x, y) \in Z^2 \text{ such that}$$

$$1 \leq t \leq 2n+1 \qquad 0 \leq x \leq \frac{t-1}{2}$$

$$0 \leq y \leq \frac{t-2x-1}{2} \qquad t-x-n-1 \leq y$$

Number of active processors at any step t
= **Number of integer points in** $P_{n,t}$



An application example

Analysis of a parallel loop ...

Computation of the Ehrhart polynomial of $P_{n,t}$

$\rightarrow$ Vertices of $P_{n,t}$:

if $1 \leq t \leq n+1 \rightarrow$ **domain (1)**:

$$(t-n-1,0) \quad (\frac{t-1}{2},0) \quad (0,t-n-1) \quad (0,\frac{t-1}{2})$$

if $n+1 \leq t \leq 2n+1 \rightarrow$ **domain (2)**:

$$(\frac{t-1}{2},0) \qquad (0,\frac{t-1}{2}) \qquad (0,0)$$

$\Rightarrow$ Ehrhart polynomial on domain (1) of the form:

$$f(n,t) = c_1 t^2 + [c_2, c_3]t + [c_4, c_5]$$

$$\text{with} \begin{cases} c_1 + c_2 + c_4 & = f(n,1) \\ 4c_1 + 2c_3 + c_5 & = f(n,2) \\ 9c_1 + 3c_2 + c_4 & = f(n,3) \\ 16c_1 + 4c_3 + c_5 & = f(n,4) \\ 25c_1 + 5c_2 + c_4 & = f(n,5) \end{cases}$$

An application example

Analysis of a parallel loop ...

Computation of $f(n,1),\ldots,f(n,5)$
of the form: $c_1n^2+[c_2,c_3]n+[c_4,c_5]$

$\rightarrow$ Counting of 5 initial values
with the scanning loop:

```
init_val:=0;
for x:=0 to (t-1)/2
    for y := max(0,t-n-x-1) to (t-2*x-1)/2
        init_val:=init_val+1;
```

For example, computation of $f(n,1)$:

$$\Rightarrow \begin{cases} c_1+c_2+c_4 & = f(1,1)=1 \\ 4c_1+2c_3+c_5 & = f(2,1)=1 \\ 9c_1+3c_2+c_4 & = f(3,1)=1 \\ 16c_1+4c_3+c_5 & = f(4,1)=1 \\ 25c_1+5c_2+c_4 & = f(5,1)=1 \end{cases}$$

$$\Rightarrow f(n,1)=1$$

An application example

Analysis of a parallel loop . . .

In the same way, $f(n,2) = 1$, $f(n,3) = 3, f(n,4) = 4, f(n,5) = 6$

$\Rightarrow$ **On domain (1)**,

$$\forall\ t \in [1..n+1],$$

$$maxpar(n,t) = \frac{1}{8}t^2 + [\frac{1}{2}, \frac{1}{4}]t + [\frac{3}{8}, 0]$$

$\Rightarrow$ In the same way, **on domain (2)**,

$$\forall\ t \in [n+1..2n+1],$$

$$maxpar(n,t) = nt - \frac{1}{2}n^2 - \frac{1}{2}n - \frac{3}{8}t^2 + \left[1, \frac{3}{4}\right]_t t + \left[\frac{3}{8}, 0\right]_t$$

An application example

Analysis of a parallel loop ...

$\Rightarrow$ Maximum number of simultaneously active processors

- On domain (1), $maxpar(n,t)$ maximum when t maximum ($t = n+1$)

$$\Rightarrow maxpar(n) = \frac{1}{8}n^2 + \left[\frac{1}{2}, \frac{3}{4}\right] + \left[\frac{3}{8}, 1\right]$$

- On domain (2), maximize $maxpar(n,t)$ relatively to t

$$\Rightarrow maxpar(n) = \frac{1}{6}n^2 + \frac{5}{6}n + [1, \frac{25}{24}, 1]$$

$\Rightarrow$ **Maximum number of simultaneously active processors for any n:**

$$maxpar = \frac{1}{6}n^2 + \frac{5}{6}n + [1, \frac{25}{24}, 1]$$

An application example

Analysis of a parallel loop ...

$\Rightarrow$ Efficiency of the virtual processor grid:

$$maxpar/np$$

$$=$$

$$\left(\frac{1}{6}n^2 + \frac{5}{6}n + [1, \frac{25}{24}, 1]\right) / \left(\frac{1}{2}n^2 + \frac{3}{2}n + 1\right)$$

$$=$$

$$\frac{1}{12}\ \frac{4n^2+20n+[24,25,24]}{n^2+3n+2}$$

$$\approx$$

$$1/3$$

At most one third of the processor grid is used

Conclusion

Method based on Ehrhart polynomials

=

Entirely automatic

- Parametric polytope definition (system of linear constraints) or system of non-linear constraints transformed to linear ones (W. Pugh)
- Parametric vertices finding algorithm (Loechner & Wilde)
- Scanning loop (Fourier-Motzkin) for some initial low values
- System of linear equations

Applications to optimize parallel programs

- Modeling as an enumeration of integer points …can be automated for several specific applications

Conclusion

Related works

- *Haghighat & Polychronopoulos*: Results more complicated (min, max)

- *Ferrante, Sarkar & Thrash*: No unions of polytopes, no symbolic answers, approximations, expensive methods.

- *Pugh*: Transformations (often intuitive, expensive) of the initial set of constraints (Presburger formulas) in order to use standard formulas for sums of powers.

- *Tawbi*: Polyhedral splitting technique, does not systematically compute exact answers.